On the relation between covariant and canonical Quantum Gravity [arXiv:gr-qc/1307.5885]

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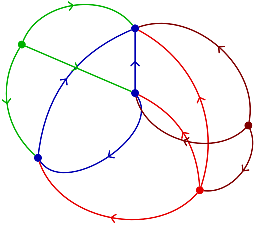
Motivation

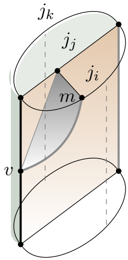
Canonical LQG



$$T_s(A) = \mathrm{Tr}[\prod_l h_l^j(A) \prod_n \iota_n]$$

$$\mathcal{H}_{kin} = \bigoplus_{\gamma \in \Sigma} \mathcal{H}_{kin,\gamma}$$

Covariant LQG

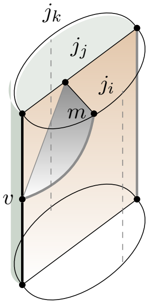


$$Z[\kappa] = \sum_c \prod_v \mathcal{A}_v \prod_f \mathcal{A}_f \times \mathcal{B}$$

$$\mathcal{H}_{\partial\kappa} \simeq \mathcal{H}_{kin,\gamma}$$

Is it possible to merge covariant and canonical LQG?

The general idea



$$\langle T_s | Z[\kappa] | T_{s'} \rangle$$

$$\partial\kappa = \gamma_s \cup \gamma_{s'}$$

Rovelli,Reisenberger,Barrett, Crane, Engle, Pereira, Livine, Freidel, Krasnov, Kaminski, Kisielowski, Lewandowski, ...

Heuristic Idea

$$\delta(H) = \int \exp(itH)\,\mathrm{d}t \leadsto \sum_{\kappa} \underbrace{Z[\kappa]}_{\text{Feynman diagrams}}$$

[Reisenberger, Rovelli]

"Physical Scalar Product"

$$\eta[T_s](T_{s'}) := \sum_{\kappa:\gamma_{s'} \to \gamma_s} \langle T_s | Z[\kappa] | T_{s'} \rangle$$

Main Results

The extension of the [Engle,Pereira,Rovelli, Livine]-[Freidel, Krasnov]-model by [Kaminski, Kisielowski, Lewandowski] enables to investigate whether spin foams can be used to construct a rigging map for any of the presently defined Hamiltonian constraint operators.
We suggest a 'rigging map' closely following the ideas of [Reisenberger, Rovelli].

In the analysis of the resulting object we are able to identify an elementary spin foam transfer matrix that allows to generate any finite foam as a finite power of the transfer matrix.

It transpires that the postulated map does not define a projector on the physical Hilbert space. However, it might be possible to construct a proper rigging map in terms of a modified transfer matrix.

Outline

1. A spin foam rigging map
   - Rigging maps and spin foams - a brief review
   - Proposal for a spin foam ‘rigging map’
   - Properties
2. Why the would-be ‘rigging map’ is not a proper rigging map
   - Time ordering
   - Factorization
   - Why η is not a rigging map
3. Discussion and outlook
   - What is going wrong?
   - Open questions

Rigging maps

How to solve the constraints of canonical QG?

Generalized solution of C

.. is a state $l \in \mathcal{D}^*_{kin}$ s.t $[C^* l](f) := l(C^\dagger f) = 0 \ \forall f \in \mathcal{D}_{kin}$

$\mathcal{D}_{kin} \subset \mathcal{H}_{kin}$ dense domain, algebraic dual $\mathcal{D}^*_{kin}$

Construction principle? What is the scalar product?

Rigging map

Given $\eta : \mathcal{H}_{kin} \to \mathcal{D}^*_{kin}$ s. t. $\langle \eta(f), \eta(f') \rangle_{phys} = [\eta(f')](f)$

Euclidean spin foam models

1. Action: $S_{BF} = \int_{\mathcal{M}} \mathrm{Tr}[(B + \frac{1}{\beta} * B) \wedge F]$ + Simplicity constraint

   $\mathcal{M}$ space-time, B bivector, F curvature of $\mathfrak{spin}(4)$-connection on $\mathcal{M}$, β Barbero-Immirzi parameter

2. Discretize: $\mathfrak{T}$ triangulation of $\mathcal{M}$ with dual 2-complex $\kappa_{\mathfrak{T}} \rightsquigarrow S_{BF}[\kappa_{\mathfrak{T}}]$

3. Quantize $\int \mathcal{DP}\, \mathrm{e}^{iS_{BF}[\kappa_{\mathfrak{T}}]} \rightsquigarrow \mathcal{A}^{BF}[\kappa_{\mathfrak{T}}, \psi_{in}, \psi_{out}] = \langle \psi_{in}, Z^{BF}[\kappa_{\mathcal{T}}]\psi_{out}\rangle$

4. Implement simplicity constraint

Result

$$\mathcal{A}^{EPRL-FK}[\kappa_{\mathfrak{T}}, \psi_{in}, \psi_{out}] = \langle \psi_{in}, Z[\kappa_{\mathfrak{T}}]\psi_{out}\rangle =$$

$$\sum_{j_f^{\pm}, \iota_e^{EPRL}} \prod_f \mathcal{A}_f \prod_{e \in \kappa_{int}} Q_e \prod_{v \in \kappa_{int}} \mathcal{A}_v(j_f^{\pm}, \iota_e^{EPRL})\, \mathcal{B}(\psi_{in}, \psi_{out})$$

$j^{\pm} = \frac{|\mathbf{1} \pm \beta|}{\mathbf{2}} j$, ι^{EPRL} intertwiner coupling j and (j^+, j^-)

Merging canonical and covariant LQG - Prerequisites

Can we construct $\eta[T_s](T_{s'}) = \sum_{\kappa:\gamma_{s'}\to\gamma_s} \langle T_s|Z[\kappa]|T_{s'}\rangle$?

Need identification of boundary states ψ_{in}, ψ_{out} with states in $\mathcal{H}_{kin}$

Canonical LQG	EPRL-FK Model (Eucl.)
Continuum theory	Discrete theory
Quantize point separating sub-algebra of Poisson algebra $\rightsquigarrow$ all possible graphs in Σ	κ dual to triangulation $\mathfrak{T}$ of $\mathcal{M}$, $\rightsquigarrow$ boundary graphs dual to triangulation of Σ
Hamiltonian $\rightsquigarrow$ 3-valent nodes	at least 4-valent nodes

Other issues: Path integral measure? Fate of observables? Implementation of simplicity constraint?

Abstract foams - The KKL-model

Spin foams - a tool to determine the physical scalar product ?

KKL-model

[Kaminski, Kisielowski, Lewandowski]

Achievements:

1. $Z[\kappa]$ defined for arbitrary 2-complexes allowing arbitrary boundary graphs $\partial\kappa$
2. For certain choice of β: $\mathcal{H}_{\partial\kappa} \simeq \mathcal{H}_{kin,\partial\kappa}$

In the following we will consider abstract foams

Abstract foam κ: p.l. homeomorphic to a 2-complex $\{f, e, v\}$ whose boundary $\partial\kappa$ is the disjoint union of closed graphs bordering κ

γ borders κ: exist 1-to-1 affine map $\gamma \times [0,1] \to \kappa$

Proposed rigging-map

Abstract equivalence

Two embedded spin nets belong to the same abstract equivalence class $[s]_A$ if they are embeddings of the same abstract spin net s_A.

A spin foam rigging map

$$\eta[T_s](T_{s'}) = \sum_{[s']_A \in N_A} \eta_{[s]_A,[s']_A} L_{[s']_A} \quad \text{with} \quad \eta_{[s]_A,[s']_A} = \sum_{\kappa_A(s_A,s'_A)} Z[\kappa_A(s_A,s'_A)]$$

$$\text{and} \quad L_{[s']_A} = \eta_{[s']_A} \sum_{\hat{s}\in[s']_A} \langle T_{\hat{s}}, \cdot \rangle$$

$$Z[\kappa_A(s_A,s'_A)] := \sum_{j_f,\iota_e} \prod_f \mathcal{A}_f \prod_{e\in\kappa_{int}} Q_e \prod_{v\in\kappa_{int}} \mathcal{A}_v(j_f,\iota_e) \prod_{l\in\partial\kappa} \delta_{j_l,j_{f_l}} \prod_{n\in\partial\kappa} \delta_{\iota_n,\iota_{e_n}}$$

[Kaminski, Kisielowski, Lewandowski], [Ding, Han, Rovelli],[Bahr, Hellmann, Kaminski, Kisielowski, Lewandowski]

Properties of $Z[\kappa]$

$$Z[\kappa_A(s_A, s'_A)] := \sum_{j_f, \iota_e} \prod_f \mathcal{A}_f \prod_{e \in \kappa_{int}} Q_e \prod_{v \in \kappa_{int}} \mathcal{A}_v(j_f, \iota_e) \prod_{l \in \partial\kappa} \delta_{j_l, j_{f_l}} \prod_{n \in \partial\kappa} \delta_{\iota_n, \iota_{e_n}}$$

[Kaminski, Kisielowski, Lewandowski], [Ding, Han, Rovelli], [Bahr, Hellmann, Kaminski, Kisielowski, Lewandowski]

Can be defined s.t. $Z[\kappa_A(s_A, s'_A)]$ is invariant if ...

- ...faces labeled by trivial representation are added or removed
- ...internal edges splitting a face are added or removed
- ...2-valent vertices are added or removed

$Z[\kappa_A](s_A, s'_A) : \mathcal{H}_{kin,\gamma(s'_A)} \to \mathcal{H}_{kin,\gamma(s_A)}$ is cylindrically consistent!

Furthermore: $\forall\ \gamma\ \exists\ \kappa^0_\gamma$ s.t. $Z[\kappa^0_\gamma] : \mathcal{H}_{kin,\gamma} \to \mathcal{H}_{kin,\gamma}$ with $Z[\kappa^0_\gamma] = \mathbb{1}_\gamma$

Minimal foams

Note

Since adding faces with $j_f = 0$, splitting faces and edges leave the amplitude/spin net function invariant one should only sum over equivalence classes in the rigging map .

Definition

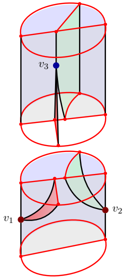
An abstract foam/graph is called minimal iff it cannot be obtained from another foam/graph by subdivisions.

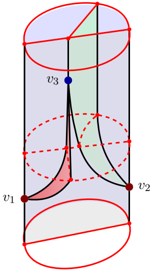
Remark

- Minimal representative is not unique . . .
- . . . but amplitude does not depend on choice

Can always fix a representative for each class and only sum over those.

Gluing



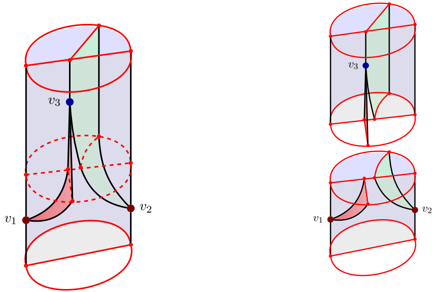


Suppose $\kappa_1 \cap \kappa_2 = \partial\kappa_1 \cap \partial\kappa_2 = \tilde{\gamma}$ then

$$\sum_{\tilde{s}(\tilde{\gamma})} Z[\kappa_1(s,\tilde{s})]\, Z[\kappa_2(\tilde{s},s')] = Z[\kappa_1 \sharp \kappa_2(s,s')]$$

Outline

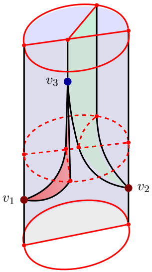
Spliting



Idea: Use this to split big complexes to get a better control on

$$\eta[T_s](T_{s'}) := \sum_{\kappa:\gamma_{s'}\to\gamma_s} \langle T_s | Z[\kappa] | T_{s'} \rangle$$

Time ordering



Definition

- $v \in \kappa_{int}$ s.t. $\exists n \in \gamma_i$ and $\exists e \in \kappa_{int}$ with $s(e) = n$ and $t(e) = v$
  $\rightsquigarrow$ vertex of first generation
- Inductively: Vertex of n^{th} generation
- Only final graph $\Rightarrow$ count backwards
- $\partial\kappa = \varnothing$ all $v \in \kappa$ of first generation

Theorem

A minimal foam κ can be uniquely split into minimal foams κ_i, called 'one time step' foams, containing only vertices of i^{th} generation with respect to the original foam $\kappa = \kappa_1 \sharp \cdots \sharp \kappa_N$ where N is the maximal generation of κ.

Sketch of the Proof I

To prove the theorem we need the following observations:

Lemma

If $e \in \kappa_{int}$ has a vertex of n^{th} generation then the other vertex of e is either of generation $n-1, n$ or $n+1$ or a vertex in a final graph.

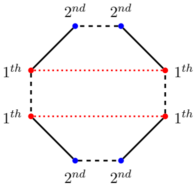
Introduce additional vertices so that the second vertex of e is either of generation $n-1, n$ or $n+1$.

Lemma

If a face has an edge e joining a vertex of n^{th} and $(n+1)^{th}$ generation then it contains at least one other edge connecting vertices of n^{th} and $(n+1)^{th}$ generation (or v of generation m and v' in final graph).

Sketch of the Proof II

Proof of the theorem



- $\mathcal{E}_{n,n+1} := \{e \in \kappa_{int} :$ joining vertices of n^{th} and $(n+1)^{th}$ generation$\}$
- Number of boundary edges $e \in \mathcal{E}_{n,n+1}$ of f is even
- Split faces until a face contains at most 2 edges in $\mathcal{E}_{1,2}$
- Split all edges in $\mathcal{E}_{1,2}$ and join new vertices by edges

$\rightsquigarrow \; \kappa = \kappa_1 \sharp \kappa'$

Repeat the procedure until $\kappa = \kappa_1 \sharp \cdots \sharp \kappa_N$

Factorization of the rigging map I

If η is a proper rigging map then ...

$$\eta[T_{s_f}](\hat{H} T_{s_i}) \stackrel{!}{=} 0 \quad \forall T_{s_i} \in \mathcal{H}_{kin}$$

$$\implies \eta[T_\varnothing](\hat{H} T_{s_i}) = \left(\sum_n (\eta^c[T_\varnothing](T_\varnothing))^n\right) \eta^c[T_\varnothing](\hat{H} T_{s_i}) \stackrel{!}{=} 0$$

where $\sum_\kappa$ in η^c is restricted to connected foams

Thus to test the rigging map it suffices to only consider connected foams!!

- But when splitting a connected κ into 'one time step' foams $\kappa_1 \sharp \cdots \sharp \kappa_n = \kappa$ then κ_i are not necessarily connected
- However, if $T_\varnothing$ is excluded and either $\gamma(s_f)$ and or $\gamma(s_i)$ are connected then any foam $\kappa_1 \sharp \cdots \sharp \kappa_n$ is connected

Test wether $\eta^c[T_{s_f}]$ annihilates H for any s_f with connected graph $\gamma(s_f)$

Factorization of the rigging map II

Spin foam transfer matrix

$\hat{K}_{\gamma,\gamma'}$ set of 'one time step' foam and $P_\gamma : \mathcal{H}_{kin} \to \mathcal{H}_{kin,\gamma}$

$$\hat{Z} := \sum_{\gamma,\gamma'} P_{\gamma'} \, [\sum_{\hat{\kappa} \in \hat{K}_{\gamma,\gamma'}} Z(\hat{\kappa})] \, P_\gamma$$

We then obtain for $\gamma(s_f)$ connected ...

$$\eta^c[T_{s_f}](T_{s_i}) = \sum_{\kappa \in K_{\gamma(s_i),\gamma(s_f)}} \langle T_{s_f}, \; Z(\kappa_1 \sharp \cdots \sharp \kappa_N) \; T_{s_i} \rangle = \sum_{N=0}^{\infty} \langle T_{s_f}, \hat{Z}^N T_{s_i} \rangle$$

Remarks

- Above factorization would lead to ordering ambiguities for disconnected foams
- $\hat{Z}$ is symmetric

A naive argument

If η would be a proper rigging map then

$$\eta^c[T_{s_f}](HT_{s_i}) = \langle T_{s_f}, \sum_{N=0}^{\infty} \hat{Z}^N HT_{s_i}\rangle \stackrel{!}{=} 0$$

But ...

$$A := \sum_{N=0}^{\infty} Z^N \;\Rightarrow\; A = \mathbb{1} + ZA$$

leads to a contradiction:

$$0 = AH = (\mathbb{1} + ZA)H = H$$

Of course, this expression is only formal as A is very likely diverging and therefore requires a regularization

Regularization of $\hat{Z}$

Cut-Off: $K_{\gamma,\gamma'}$ is infinite, therefore introduce cut-off (J, N_f, N_e)

Weight: Suppose there exist a weight $\omega(\kappa \sharp \kappa') = \omega(\kappa)\omega(\kappa')$ s.t.
$Z' := \sum_{\hat{\kappa}\in\hat{K}_{\gamma,\gamma'}} \omega(\hat{\kappa}) Z[\hat{\kappa}]$ has finite norm

Even if Z' is now densely defined $(Z')^n$ is not necessarily densely defined.

Recall: Z is symmetric and so is Z'

Suppose: Z' can be extended to an self-adjoint operator with projection valued measure E and let

Then: Z' acts on the states $\psi_q := \int_{-q}^{q} dE(\lambda)\, \psi$, $\psi \in \mathcal{H}_{kin}$, by multiplication with $\lambda \in [-q, q]$ where $0 < q < 1$

But: $A\psi_q = \int_{-q}^{q} dE(\lambda)\, (1-\lambda)^{-1}\, \psi \rightsquigarrow A = \mathbb{1} + ZA$ ↯

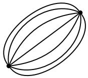
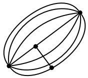
Outline

What is going wrong?

Option I:	Z is itself a projector, e.g. BF-theory
Option II:	Vertex amplitude too local
Option III:	Necessity to restrict to one Plebanski sector?
Option IV:	Wrong assumption on the weight!

What is going wrong?

Option I: Z is itself a projector, e.g. BF-theory

1. It can be shown that for $\beta = 1$ the operator $\sum_{\kappa \in K_{\text{Melon}}} Z[\kappa]$ annihilates the Euclidean constraint. Here, $\kappa \in K_{\text{Melon}}$ are foams with just one internal vertex and boundary graphs of the form:  and 

   [Alesci, Thiemann, A.Z.]

2. Z can be interpreted as an object obtained from some sort of coarse graining. In [Dittrich, Hellmann, Kaminski] a similar object was derived for holonomy spin foam models via coarse graining. For BF-theory this already incorporates the dynamics.

What is going wrong?

Option II: Vertex amplitude too local

The would-be rigging map can be factorized since foams can be split, that is, $\kappa = \kappa_1 \sharp \cdots \sharp \kappa_N$

Suppose $\mathcal{A}_v Q_e(v, v')\mathcal{A}_{v'} \not\propto \mathcal{A}_v Q_e(v, m)\mathcal{A}_m Q_e(m, v')\mathcal{A}_{v'}$ then splitting is no longer possible.

In this sense: vertex amplitude too local

Other hints into this direction: E.g. [Dittrich], [Hellmann, Kaminski], . . .

What is going wrong?

Option III: Necessity to restrict to one Plebanski sector?

$$2\pi\delta(C) = \lim_{T\to\infty}\int_{-T}^{T} e^{itC} = \lim_{T\to\infty}\int_{0}^{T}\left[e^{itC} + e^{-itC}\right]$$

$$= \lim_{T\to\infty}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{T}{n}\underbrace{\left[e^{iCT/n}\right]^k}_{U^k} + \underbrace{\left[e^{-iCT/n}\right]^k}_{(U^\dagger)^k}$$

But: $\hat{Z}$ is symmetric $\rightsquigarrow$ $\hat{Z} = U + U^\dagger$

$$\Rightarrow \sum_{k\in\mathbb{N}} Z^k = \sum_{k\in\mathbb{N}}(U + U^\dagger)^k \neq \delta(H)$$

A similar problem occurs in the asymptotic expansion.

Can it be solved by the proper vertex proposal [Engle]?

What is going wrong?

Option IV: Wrong assumption on the weight!

We assumed that Z can be regularized by introducing a weight ω that obeys $\omega(\kappa_1)\omega(\kappa_2) = \omega(\kappa_1 \sharp \kappa_2)$. Is that sensible?

1. Regularization might require symmetry factors that destroy splitting:

$$\delta(x) = \int \mathrm{d}k\, e^{ikx} = \int \mathrm{d}k (\sum_n \frac{i^n}{n!} (kx)^n)$$

For example in standard GFT $\frac{\lambda^{|\gamma^{(0)}|}}{|\mathrm{sym}(\gamma)|}$

2. Spin foam amplitudes suffer from bubble divergencies. However, if foams with bubbles are excluded then gluing might no longer be possible, since even if κ_1, κ_2 contain no bubbles, $\kappa_1 \sharp \kappa_2$ might have bubbles.

Open questions

Regularization

- Explore regularization methods of generalized GFT [Oriti]
- Explore latest results on regularization of the EPRL-model [Bonzom,Carrozza,Oriti, Puchta, Riello, Rovelli, Smerlak,. . .]
- Connection to coarse graining [Dittrich, . . .]?

Proper vertex

Generalize proper vertex proposal [Engle] to n-valent vertices and test wether the modified rigging map still has the same problems.

Technical aspects

- Single time step $\rightsquigarrow$ better control on the sum
- Formalism to design single time step foams; [Kisielowski, Lewandowski, Puchta]

Thank you for your attention!

Contents