Curvature

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Outline

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Applications to effective theories of connections

- Reconstruction of the bundle from a discrete set of data
- Approximate localization of the connection modulo gauge
- Coarse graining
- Regularization

Applications to geometry and topology of fiber bundles

- Reconstruction
- Non abelian Stokes formula
- Invariants of principal fiber bundles

Outline

Notation

- M a smooth manifold
- $\pi = (E, \pi, M)$ a smooth principal G bundle
- $\mathcal{A}_\pi$ the space of smooth connections
- $\mathcal{A}/\mathcal{G}_{\star,\pi}$ connections modulo gauge that are the identity on $\pi^{-1}(\star)$, for $\star \in M$ a base point
- $\mathcal{L}(M, \star)$ the group of $\star \in M$ based loops modulo reparametrization and retracing
- $\mathcal{L}^0(M, \star) \subset \mathcal{L}(M, \star)$ subgroup of contractible loops

Notation

Given $[l] \in \mathcal{L}(M, \star)$ and $[A] \in \mathcal{A}/\mathcal{G}_{\star,\pi}$

$$\mathrm{Hol}(l, A) : \pi^{-1}(\star) \to \pi^{-1}(\star)$$

After identifying $\pi^{-1}(\star)$ with G we can write

$$\mathrm{Hol}(l, A) \in G$$

Preliminaries

- S contractible surface with piecewise smooth boundary
- γ pw smooth path starting at $\star$ and finishing at $x \in \partial S$
- $(S, \gamma) \longmapsto [\partial S_\gamma] = [\gamma^{-1} \circ \partial S \circ \gamma] \in \mathcal{L}^0(M, \star)$

We consider $\{S_t\}_{t \in [0,1]}$ such that:
(i) $S_t \subset S$
(ii) $[\partial S_{t,\gamma}] \in \mathcal{L}^0(M, \star)$
(iii) $[\partial S_{t=0,\gamma}] = [\star]$ and $[\partial S_{t=1,\gamma}] = [\partial S_\gamma]$

Preliminaries

Given $[A] \in \mathcal{A}/\mathcal{G}_{\star,\pi}$, the family of surfaces determines

$$\begin{aligned} c : [0,1] &\rightarrow G \\ t &\mapsto \mathrm{Hol}(\partial S_{t,\gamma}, A) \end{aligned}$$

a curve in G with $c(0) = \mathrm{id}$, $c(1) = \mathrm{Hol}(\partial S_{\gamma}, A)$.

$\tilde{c}$ unique lifting curve in $Lie(G)$ such that

$$\exp \tilde{c}(t) = c(t) \quad \text{and} \quad \tilde{c}(0) = 0$$

Definition

Definition (Curvature)

$$F_{S_\gamma}(A) = \tilde{c}(1)$$

This definition is independent of the choice $\{S_t\}_{t\in[0,1]}$:
S being contractible implies that another choice $\{S'_t\}_{t\in[0,1]}$ would result in a curve in G $c' \sim c$ and thus $\tilde{c}'(1) = \tilde{c}(1)$.

Clearly $\exp F_{S_\gamma}(A) = \mathrm{Hol}(\partial S_\gamma, A)$

Example

$M = S^2$, $G = SO(2) \approx S^1$, π assoc. to the frame bundle,
$[A]$ induced by the Levi-Civita connection of $S^2_{r=1}$
$S \subset S^2$ covering all of S^2 except for a small portion

$$F_S(A) = \text{Area}(S) \in \mathbb{R}$$

$$\mathrm{Hol}(\partial S, A) = \text{Angle}(S) \in SO(2)$$

Calculation according to general definition:

- $c(t) = \text{Angle}(S_t)$ winds more than once on $SO(2) \approx S^1$
- $\tilde{c}(t)$ starts at 0 and ends at $F_S(A) = \text{Area}(S) \in \mathbb{R}$

Lifted Parallel Transport

Definition (Lifted Parallel Transport)

(with respect to $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$)

$$\tilde{P}_\gamma(A) = \text{ end point of lifted curve } \in Lie(G)$$

where $A \in \mathcal{A}_\pi$, $\gamma \subset U_\alpha$ is a piecewise smooth path, and $\{\gamma_t\}_{t\in[0,1]} \cdot st\cdot\ \gamma_t \subset \gamma$ subpath, $\gamma_t(0) = \gamma(0)$ and $\gamma_{t=1} = \gamma$.

$$P(t) = P_{\gamma_t}(A) \in G \text{ curve of parallel transport along } \gamma$$

Remark: can substitute ingredients $\phi_\alpha \longleftrightarrow A_0 \in \mathcal{A}_\pi$

Stokes Formula

Reminder of calculus on manifolds (Arnold's presentation)

$$\int_S d\omega \doteq \int_{\partial S} \omega$$

This formula defines $d\omega$.

Theorem

If the local coordinate expression for ω is $\omega(x) = \omega_i(x)dx^i$, then

$$d\omega(x) = d\omega_i(x) \wedge dx^i = \frac{\partial \omega}{\partial x^j}(x)dx^j \wedge dx^i$$

Non abelian Stokes Formula

In collaboration with A. Soto-Posada

Definition (Non abelian Stokes Formula)

(with respect to a local trivialization)

$$F_{S_\gamma}(A) \doteq \tilde{P}_{\partial S_\gamma}(A)$$

Can be proven to agree with previous definition

Structural equation

In collaboration with A. Soto-Posada

Let $S(v, w)$ denote the ”curved parallelogram” determined by the vectors $v, w \in T_xM$ and the chart associated to U_α.

Theorem (Structural equation)

The Lie algebra valued two form

$$F(v_x, w_x) \doteq \mathrm{Ad}_{P_\gamma^{-1}(A)} \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} F_{S(\epsilon v, \epsilon w)_\gamma}(A)$$

can be written in terms of the connection one form as

$$F_{ab}(x) = (dA)_{ab} + [A_a, A_b]$$

Outline

Effective theories of connections

Let $\phi : |\Delta| \to M$ be a triangulation such that

$\mathcal{L}(L, \star) \subset \mathcal{L}(M, \star)$ the subgroup of loops"that fit in" $L = \phi((\mathrm{Sd}|\Delta|)^{(1)})$

$\phi((\mathrm{Sd}|\Delta|)^2)$ is an array of surfaces with boundary

We will need an assignment of a path to each simplex $\tau \in \phi|\Delta|$

$\tau \mapsto \gamma_\tau \subset L$ path starting at $\star$ and finishing at $\mathrm{bar}\tau$

Then $\mathcal{L}^0(L, \star)$ is generated by $< [\partial\sigma_\gamma] >_{\sigma \in \phi((\mathrm{Sd}|\Delta|)^2)}$

Effective theories of connections

Given $[A] \in \mathcal{A}/\mathcal{G}_{\star,\pi}$ consider the families of curvature evaluations

$$\breve{\omega}(A) = \{F_{\partial\sigma_\gamma}(A) \in Lie(G)\}_{\sigma \in \check{I} \subset \phi((\mathrm{Sd}|\Delta|)^2)} \in \check{\Omega}_\Delta$$

$$\omega(A) = \{F_{\partial\sigma_\gamma}(A) \in Lie(G)\}_{\sigma \in I_\pi \subset \phi((\mathrm{Sd}|\Delta|)^2)} \in \Omega_{\Delta\pi}$$

Data for effective theories of connections over $M = S^d$

Characterization

Theorem (Characterization)

Among the principal fiber bundles with base space S^d and structure group G, the bundle $\pi = (E, \pi, S^d)$ is characterized by the data

$$\check{\omega}(A) \in \check{\Omega}_\Delta$$

in the sense that any other bundle with a connection that induces the same data is an equivalent bundle.

Approximate Localization

Theorem (Approximate Localization)

$$\textit{Any two } [A_1], [A_2] \in F_{\Delta}^{-1}(\omega) \subset \mathcal{A}/\mathcal{G}_{\star,\pi}$$

are related by $|\Delta|$*-local deformations in the sense that they can be deformed by* $|\Delta|$*-local transformations to*

$$[A]_{\Delta}(\omega) \in \mathcal{A}/\mathcal{G}_{\star,\pi,\Delta}$$

where $\mathcal{A}/\mathcal{G}_{\star,\pi,\Delta}$ *is a space of flat connections with conical singularities at* $\phi((\mathrm{Sd}|\Delta|)^{(2)})$.

- To extend to general M instead of S^d non contractible loops need to be considered
- Recall / compare these theorems to the Barrett-Kobayashi reconstruction theorem

Illustration

All $U(1)$ bundles over S^2 are distinguished by data on $\check{\Omega}_\Delta$.

$$\text{Consider } S^2 = D_N \cup D_S \quad , \quad Eq = D_N \cap D_S \approx S^1 .$$

Every bundle over a disc is trivial, thus

$$T_{SN} : Eq \to U(1) \text{ encodes bundle str.}$$

Given a connection one can construct trivializations over the discs and

$$T_{SN}(\theta) = \mathrm{Hol}(l(\star = N, 0 \in Eq, S, \theta \in Eq, N); A).$$

Illustration

If $\phi : |\Delta| \to S^2$ triangulates D_N and D_S, then $\breve{\omega}(A) \in \check{\Omega}_\Delta$ characterizes the homotopy type of T_{SN}.

Two bundles with homotopic $T_{SN} : Eq \to U(1)$ are equivalent.

Coarse Graining

Consider $\phi : |\Delta| \to M$, and $\phi' : |\Delta'| \to M$ such that

$$\tau \in \phi(\mathrm{Sd}|\Delta|) \Rightarrow \tau = \cup \tau_i' \text{ with } \tau_i' \in \phi'(\mathrm{Sd}|\Delta'|)$$

In particular, $\sigma \in \phi((\mathrm{Sd}|\Delta|)^2) \Rightarrow \sigma = \sum \sigma'^i$ as 2-chains
If we use as finer triangulation $\phi : \mathrm{Sd}|\Delta| \to M$
it is easy to choose paths $\gamma(\sigma'^i)$ that let us write

$$\partial \sigma_\gamma = \partial \sigma_\gamma'^6 \circ \ldots \circ \partial \sigma_\gamma'^1$$

Coarse Graining

The coarse graining of curvature functions follows from

Theorem (Coarse Graining)

$$F_{\sigma_\gamma}(A) = F_{\sigma'^{6}_{\gamma}}(A)\tilde{\circ}\ldots\tilde{\circ}F_{\sigma'^{1}_{\gamma}}(A)$$

where $v_2 \tilde{\circ} v_1 = \tilde{c}_{v_1}^{v_2}(1)$ *is the final point of the lift of a curve* c *in* G *starting at* id *and finishing at* $\exp v_2$ *using as starting point* $v_1 \in Lie(G)$*.*

Remark: F_{σ_γ} is better than $\mathrm{Hol}_{\partial\sigma_\gamma}$ as macroscopic observable

Example

Consider a bundle over $\sigma = \sum_{i=1}^{6} \sigma'^{i}$ with $G = U(1)$, and two connections A, A' such that $F_{\sigma^i}(A) = \pi/3$, $F_{\sigma^i}(A') = 0$. Then

$$F_\sigma(A) = 2\pi \in \mathbb{R} \quad , \quad F_\sigma(A') = 0 \in \mathbb{R}$$

while

$$\mathrm{Hol}_{\partial\sigma}(A) = \mathrm{Hol}_{\partial\sigma}(A') = \mathrm{id} \in U(1)$$

Regularization

Exemplary case: Euler character ($\dim M = 2$, $G = U(1)$)
Given $[A] \in \mathcal{A}/\mathcal{G}_{\star,\pi}$ consider the function

$$\int_M F \mapsto \sum_\sigma F_\sigma(A) = eu(\pi)$$

Invariants of principal fiber bundles

Generalizations of the one given above to

- non abelian groups
- higher dimensional manifolds

Applications

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- Regularization
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Comments for QFT

$$\exp F_{S_\gamma}(A) = \mathrm{Hol}(\partial S_\gamma, A)$$

$\Rightarrow$ An effective theory of connections based on holonomies models a **compactified** version of the system

- Standard LGT on a given lattice ant a given value of the coupling constant
- Information in “shadow” states on a single graph labeled by irreducible representations
- Standard spin foam models on a given discretization of spacetime

One has to study if in the continuum limit of the RG the compactification is significative.

Comments for QFT

There are models for gravity based on curvature:

- First order Regge calculus (Regge, Barrett)
- Freidel-Krasnov spin foam model
  (semiclassical study by Conrady and Freidel)
- ...

Thank you!