Deriving LQC Dynamics from Diffeomorphism Invariance

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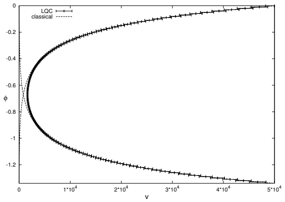
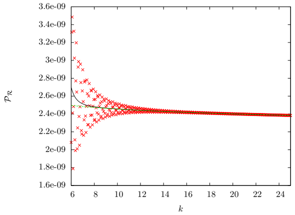
Motivation

- Loop Quantum Cosmology - quantum gravity in simplified symmetry-reduced setting
- Cosmology provides arena for testing predictions of quantum gravity
- Has seen a lot of development



Ashtekar, Pawlowski, Singh, PRD74, 084003 (2006)



Agullo, Ashtekar, Nelson, CQG30, 085014 (2013)

- How robust are these results?

Uniqueness results

- There are choices in the quantization procedure
- In the full theory seminal work by Lewandowski, Okolow, Sahlmann, Thiemann (2006) used diffeomorphism invariance to select unique representation of quantum algebra
- In symmetry-reduced setting almost all diffeomorphism symmetry is fixed except for *residual diffeos*
- Ashtekar, Campiglia (2012), Engle, Hanusch (2017), Engle, Hanusch, Thiemann (2017) used residual diffeomorphism invariance to select unique representation of reduced quantum algebra
- Can we use physical principles to also select unique dynamics?


YES,
WE
CAN

Main Result

- Obtain Bianchi I Hamiltonian proposed by Ashtekar, Wilson-Ewing (2009) using invariance under residual diffeomorphisms:
  - Volume-preserving dilations
  - Parity transformations
  - Reflections

  as well as a certain minimality principle and a planar loops assumption
- Obtain isotropic Hamiltonian proposed by Ashtekar, Pawlowski, Singh (2006) by projecting down from Bianchi I to isotropic model *without need for planar loops assumption*

Classical Bianchi I model

- Bianchi I model

$$ds^2 = -N^2(t)dt^2 + a_1^2(t)dx_1^2 + a_2^2(t)dx_2^2 + a_3^2(t)dx_3^2,$$

- Introduce fiducial cell $\mathcal{V}$ adapted to fiducial triads $\mathring{e}_i^a$ with side lengths L_1, L_2, L_3 and volume V_o, fiducial metric $\mathring{q}_{ab}$
- Basic variables c_i, p^i

$$A_a^i = c^i (L^i)^{-1} \mathring{e}_a^i \qquad E_i^a = p_i L_i V_o^{-1} \sqrt{\mathring{q}} \mathring{e}_i^a$$

- Poisson bracket

$$\{c^i, p_j\} = 8\pi G \gamma \delta_j^i,$$

Classical Hamiltonian constraint

- Hamiltonian constraint

$$C_H = \int_{\mathcal{V}} N\mathcal{H}\; d^3x,$$

where the Hamiltonian density $\mathcal{H}$ is

$$\mathcal{H} = \frac{E^a_i E^b_j}{16\pi G\sqrt{|q|}}(\epsilon^{ij}{}_k F^k_{ab} - 2(1+\gamma^2)e^{ci}e^{dj}K_{c[a}K_{b]d}).$$

- Assume the lapse $N(v)$ to be a function of the volume $v := \sqrt{|p_1 p_2 p_3|}$ only, with the form $N(v) = v^n$
- Integrating over the fiducial cell we then obtain the constraint

$$C_H = -\frac{1}{8\pi G\gamma^2} v^{n-1}(p_1 p_2 c_1 c_2 + p_1 p_3 c_1 c_3 + p_2 p_3 c_2 c_3).$$

Ansatz

- Hilbert space: almost periodic functions on $\mathbb{R}^3$.
- Require $\hat{H}$ to preserve this Hilbert space

$$\hat{H}|\vec{p}\rangle = \sum_i g_i(\vec{p})|\vec{F}_i(\vec{p})\rangle$$

- Define translation operator:

$$T_F|\vec{p}\rangle := |\vec{F}(\vec{p})\rangle$$

- Impose self-adjointness. Can rewrite $\hat{H}$ as

$$\hat{H} = \sum_i \left(T_{F_i}\, g_i(\vec{p}) + \overline{g_i(\vec{p})}\, T_{F_i}^{\dagger} \right)$$

- Want to take the classical limit in the state-independent way
- Assume $\vec{F}_i$ is generated as the flow, evaluated at unit time, of some vector field $8\pi\gamma G\hbar \vec{f}_i(\vec{p}) \cdot \nabla$ on $\mathbb{R}^3$.
- Get

$$\hat{H} = \sum_i \left(\widehat{e^{i\vec{f}_i(\vec{p})\cdot\vec{c}}}\ g_i(\vec{p}) + \overline{g_i(\vec{p})}\ \widehat{e^{-i\vec{f}_i(\vec{p})\cdot\vec{c}}} \right)$$

Invariance under volume-preserving positive rescalings

- Volume-preserving positive rescalings $\Lambda(\vec{\lambda})$, with $\lambda_1 + \lambda_2 + \lambda_3 = 0$, act on the variables c_i, p^i:

$$\Lambda(\vec{\lambda})p^i = e^{-\lambda_i} p^i \qquad \Lambda(\vec{\lambda})c_i = e^{\lambda_i} c_i .$$

- The invariance leads to condition:

$$e^{\lambda_k} f_i^k (e^{-\lambda_1} p_1, e^{-\lambda_2} p_2, e^{\lambda_1+\lambda_2} p_3) = f_i^k(\vec{p}) \qquad g_i(e^{-\lambda_1} p_1, e^{-\lambda_2} p_2, e^{\lambda_1+\lambda_2} p_3) = g_i(\vec{p}).$$

- Obtain

$$f_i^k(\vec{p}) = p^k \tilde{f}_i^k(v, \overrightarrow{\mathrm{sgn}\, p}) \qquad g_i(\vec{p}) = g_i(v, \overrightarrow{\mathrm{sgn}\, p}),$$

where $\overrightarrow{\mathrm{sgn}\, p} = (\mathrm{sgn} p_1, \mathrm{sgn} p_2, \mathrm{sgn} p_3)$

- Therefore,

$$\hat{H} = \sum_{i=1}^{N} \left(e^{i \sum_k \tilde{f}_i^k (v, \overrightarrow{\mathrm{sgn}\, p}) p^k c_k} g_i(v, \overrightarrow{\mathrm{sgn}\, p}) + \mathrm{h.c.} \right) .$$

Invariance under parity

- Invariance under parity implies that
  - Either $\tilde{f}_i^k$, g_i independent of $\mathrm{sgn} p$
  - Or $\hat{H}$ includes all the terms generated by parity so that for example

$$\hat{H} = \sum_i \Big(e^{i\tilde{f}_i^k(v,-\mathrm{sgn}p_1,\mathrm{sgn}p_2,\mathrm{sgn}p_3)p^k c_k} g_i(v,-\mathrm{sgn}p_1,\mathrm{sgn}p_2,\mathrm{sgn}p_3) +$$

$$+ e^{i\tilde{f}_i^k(v,\mathrm{sgn}p_1,\mathrm{sgn}p_2,\mathrm{sgn}p_3)p^k c_k} g_i(v,\mathrm{sgn}p_1,\mathrm{sgn}p_2,\mathrm{sgn}p_3) + \text{rest of terms}\Big) =$$

$$= \sum_i \Big(e^{i\tilde{f}_i^k(v,-1,\mathrm{sgn}p_2,\mathrm{sgn}p_3)p^k c_k} g_i(v,-1,\mathrm{sgn}p_2,\mathrm{sgn}p_3) +$$

$$+ e^{i\tilde{f}_i^k(v,1,\mathrm{sgn}p_2,\mathrm{sgn}p_3)p^k c_k} g_i(v,1,\mathrm{sgn}p_2,\mathrm{sgn}p_3) + \text{rest of terms}\Big).$$

- Parity invariance leads to

$$\hat{H} = \sum_i \Big(e^{i\sum_k \tilde{f}_i^k(v)p^k c_k} g_i(v) + \text{h.c.} \Big).$$

Invariance under reflections

- Reflections about the $x = y, x = z$ or $y = z$ planes and combinations thereof act on c_i, p^i like permutations of labels
- $\hat{H}$ is invariant if includes all the terms generated by such permutations

$$\hat{H} = \sum_i \sum_{\sigma \in S_3} \left(e^{i \sum_k (\sigma \tilde{f}_i)^k(v) p^k c_k} g_i(v) + \text{h.c.} \right) .$$

Imposition of the classical limit

- Require that the Hamiltonian reduces to the classical constraint in the classical limit
- Introduce a classicality parameter and consider the classical analogue of $\hat{H}$

$$H = \sum_i \sum_{\sigma \in S_3} \left(e^{i \sum_k (\sigma \tilde{f}_i)^k (v, \ell_p) p^k c_k} g_i(v, \ell_p) + \text{c.c.} \right)$$

- Require that the only length is the Planck length

$$\tilde{f}_i^k(v, \ell_p) = \frac{1}{\ell_p^2} \tilde{h}_i^k \left(\frac{\ell_p^3}{v} \right) = \frac{1}{\ell_p^2} \left(\tilde{h}_i^k(0) + (\tilde{h}_i^k)'(0) \frac{\ell_p^3}{v} + \mathcal{O}\left(\frac{\ell_p^6}{v^2} \right) \right)$$

- Also use dimensional arguments for g_i

$$g_i(v, \ell_p) = \frac{\ell_p^{3n+1}}{G} \left(\sum_{j=j_0}^{\infty} \tilde{B}_i^j \frac{\ell_p^{3j}}{v^j} \right)$$

Classical limit

- Let $\tilde{h}_i^k(0) = 0$ for all $i \in I$. Then matching the classical limit leads to

$$H = \sum_{i \in I} \sum_{\sigma \in S_3} \left(\frac{\ell_p^{3n+1}}{G} \left(\sum_{j=j_0}^{\infty} \tilde{B}_i^j \frac{\ell_p^{3j}}{v^j} \right) e^{i\left(\frac{\ell_p}{v} \sum_k (\sigma \tilde{A}_i)^k p_k c_k + \mathcal{O}(\ell_p^4)\right)} + \text{c.c.} \right) + \mathcal{O}(\ell_p^{\epsilon})$$

- Can simplify

$$H = \sum_{i=1}^{N'} \frac{\ell_p^{3n+1}}{G} \left(\sum_{j=j_0}^{\infty} B_i^j \frac{\ell_p^{3j}}{v^j} \right) e^{i\left(\frac{\ell_p}{v} \sum_k A_i^k p_k c_k + \mathcal{O}(\ell_p^4)\right)} + \mathcal{O}(\ell_p^{\epsilon})$$

- Combine exponentials (for simplification) so that

$$\vec{A}_i = \vec{A}_j \text{ implies } i = j.$$

- Expand the exponentials and match to the classical limit

Conditions for matching classical limit

- Conditions obtained by matching

$$\sum_i \mathrm{Re}\, B_i = 0$$

$$\sum_i A_{ij} \mathrm{Im}\, B_i = 0$$

$$\sum_i A_{ij} \left(\mathrm{Re}\, B_i\right) A_{ik} = M_{jk}.$$

where

$$M := \frac{1}{8\pi\gamma^2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

- Minimality Assumption: smallest number of terms so that all the conditions on $\hat{H}$ are satisfied
- Consider up to 12 rows in matrix A which corresponds to AW

Possibilities

- Eight rows

$\begin{pmatrix} a_1 & a_1 & b_1 \end{pmatrix}$
2 permutations
3 reflections
$\begin{pmatrix} a_2 & a_2 & a_2 \end{pmatrix}$
1 reflection

$\begin{pmatrix} a_1 & -a_1 & 0 \end{pmatrix}$
2 permutations
3 reflections
$\begin{pmatrix} a_2 & a_2 & a_2 \end{pmatrix}$
1 reflection

- Ten rows

$\begin{pmatrix} a_1 & a_1 & b_1 \end{pmatrix}$
2 permutations
3 reflections
$\begin{pmatrix} a_2 & a_2 & a_2 \end{pmatrix}$
1 reflection
$\begin{pmatrix} a_3 & a_3 & a_3 \end{pmatrix}$
1 reflection

$\begin{pmatrix} a_1 & -a_1 & 0 \end{pmatrix}$
2 permutations
3 reflections
$\begin{pmatrix} a_2 & a_2 & a_2 \end{pmatrix}$
1 reflection
$\begin{pmatrix} a_3 & a_3 & a_3 \end{pmatrix}$
1 reflection

Possibilities (cont.)

- Twelve rows

$\begin{pmatrix} a_1 & a_1 & b_1 \end{pmatrix}$
2 permutations
3 reflections
$\begin{pmatrix} a_2 & a_2 & a_2 \end{pmatrix}$
1 reflection
$\begin{pmatrix} a_3 & a_3 & a_3 \end{pmatrix}$
1 reflection
$\begin{pmatrix} a_4 & a_4 & a_4 \end{pmatrix}$
1 reflection

$\begin{pmatrix} a_1 & -a_1 & 0 \end{pmatrix}$
2 permutations
3 reflections
$\begin{pmatrix} a_2 & a_2 & a_2 \end{pmatrix}$
1 reflection
$\begin{pmatrix} a_3 & a_3 & a_3 \end{pmatrix}$
1 reflection
$\begin{pmatrix} a_4 & a_4 & a_4 \end{pmatrix}$
1 reflection

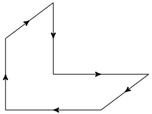
Possibilities (cont.)

- Twelve rows (plus some conditions on a_1, b_1, a_2, b_2)
  $\begin{pmatrix} a_1 & a_1 & b_1 \end{pmatrix}$
  2 permutations
  3 reflections
  $\begin{pmatrix} a_2 & a_2 & b_2 \end{pmatrix}$
  2 permutations
  3 reflections

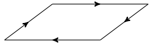
- Twelve rows
  $\begin{pmatrix} a_1 & -a_1 & 0 \end{pmatrix}$
  2 permutations
  3 reflections
  $\begin{pmatrix} a_2 & a_2 & b_2 \end{pmatrix}$
  2 permutations
  3 reflections

Selection of AW Hamiltonian

- Impose planar loops



depends on $e^{i\frac{\ell_p}{v}A^k p^k c_k} + \mathcal{O}(\ell_p^\epsilon)$
with all A^k non-zero



one of A^k is zero

- Matrix A is ($\Delta\ell_p^2$ the area gap)

$$\begin{pmatrix} \sqrt{\Delta} & -\sqrt{\Delta} & 0 \\ \sqrt{\Delta} & \sqrt{\Delta} & 0 \end{pmatrix}$$

- Obtain AW Hamiltonian

$$H_{AW} = \frac{1}{32\pi G\gamma^2\Delta\ell_p^2} v^2 \Big(e^{i\left(\frac{\sqrt{\Delta}\ell_p}{v}(p_1c_1+p_2c_2)\right)} - e^{i\left(\frac{\sqrt{\Delta}\ell_p}{v}(p_1c_1-p_2c_2)\right)} + e^{i\left(\frac{\sqrt{\Delta}\ell_p}{v}(p_2c_2+p_3c_3)\right)} - $$
$$- e^{i\left(\frac{\sqrt{\Delta}\ell_p}{v}(p_2c_2-p_3c_3)\right)} + e^{i\left(\frac{\sqrt{\Delta}\ell_p}{v}(p_1c_1+p_3c_3)\right)} - e^{i\left(\frac{\sqrt{\Delta}\ell_p}{v}(p_1c_1-p_3c_3)\right)} + \text{h.c.}\Big) + \mathcal{O}(\ell_p^2)$$

Selection of isotropic Hamiltonian

- Do not need planar loops assumption
- Use AW projector from Bianchi I states to isotropic states:

$$(\hat{\mathbb{P}}\Psi)(v) := \sum_{p_1,p_2} \Psi(p_1, p_2, v) \equiv \psi(v).$$

- Use the minimum number of terms to obtain $A = \begin{pmatrix} 0 \\ a \\ -a \end{pmatrix}$
- Obtain the APS Hamiltonian

$$H_{APS} = k\ell_p^{-2} v \left(1 + e^{i\left(\frac{\ell_p}{v} 2\sqrt{\Delta} pc\right)} + e^{-i\left(\frac{\ell_p}{v} 2\sqrt{\Delta} pc\right)}\right) + \mathcal{O}(\ell_p^2)$$

Summary

- Derived Bianchi I Hamiltonian from residual diffeomorphism invariance and minimality principle as well as a planar loops assumption.
- Obtained isotropic Hamiltonian without recourse to the planar loops assumption
- Our results increase confidence in phenomenological predictions of LQC as coming from the use of the holonomy-flux algebra
- By relaxing the minimality assumption have parametrization of ambiguities in the quantum Hamiltonian