

# Relational observables and cosmological perturbation theory

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References:

*A perturbative approach to Dirac observables and their space-time algebra*, Bianca Dittrich, jt, CQG **24** No. 4, 757-783, [gr-qc/0610060]

*Gauge invariant perturbations around symmetry reduced sectors of general relativity: applications to cosmology*, Bianca Dittrich, jt, CQG **24** No. 18, 4543-4585, [gr-qc/0702093]

## Plan of the talk

- Motivation
- Relational observables
  - General idea and framework
  - A perturbative approach
- Cosmological perturbation theory
  - Standard approach
  - Relational approach
- An example: Bianchi-I as perturbation around FRW
- Conclusions and outlook

## Motivation

### Relational observables in cosmology, why?

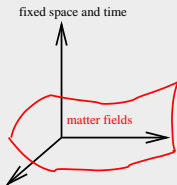
- GR is a **background-independent** theory, gauge invariant dynamics can be formulated using **relational observables**
- cosmology is the subarea of GR that deals with solutions that are close to **homogeneous** and **isotropic** on large scales
- using relational observables in cosmological perturbation theory has several advantages:
  - gauge invariance
  - background independence
  - interpretation of the results
- can reproduce standard results of linearized theory
- first cosmological perturbation scheme that keeps validity in arbitrary orders (e.g. **backreaction effects**)

## Relational observables: General idea

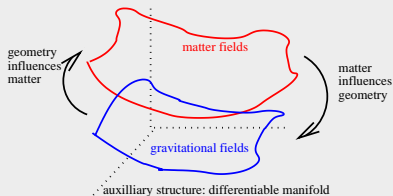
### heuristic idea

- classical GR is a **gauge theory** (4d-diffeomorphism group)
- classical GR: theory on a differentiable manifold  $M$ , not on a metric one
- $\phi(x), x \in M$  is **NOT** an observable
- Observables: **relations between dynamical fields!**  
e.g.  $\phi(x) \leftrightarrow g_{\mu\nu}(x)$

### non-relational



### relational



## Relational observables: General idea

### Hamiltonian constrained theories

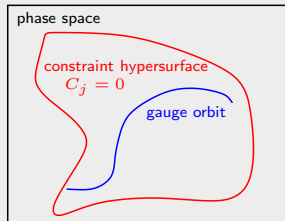
- phase space  $\{q_i, p_j\} = \delta_{ij}$ ,  $i, j \in \mathcal{I}$
- first class constraint algebra  $\{C_i, C_j\} = f_{ijk} C_k$
- constraints generate gauge orbits in phase space:
 

**gauge orbit:**  $\alpha_{\beta_l C_l}(\mathbf{x}) := \sum_0^{\infty} \frac{1}{k!} \{\mathbf{x}, \beta_l C_l\}_{(k)}$      $\mathbf{x}$ : arbitrary phase space point

$\{\cdot, \beta_l C_l\}_{(k)} := \{\{\cdot, \beta_l C_l\}_{(k-1)}, \beta_m C_m\}$  (iterated Poisson brackets)
- observables have to be constants along the **gauge orbit**

$\Rightarrow O$  is a **DIRAC OBSERVABLE**

$\Leftrightarrow \{O, C_j\} \simeq 0 \quad \forall j$



## Relational observables: General idea

### Gauge invariant observables for systems with one constraint [Rovelli '90a]

- consider “reparametrized free particle” in 1 spatial dimension

$$1 = \{x, p_x\} = \{T, p_T\}$$

$$C = p_T + \frac{1}{2m}p_x^2$$

- $T$  or  $x$  are **NOT** observables
- But: we can construct a Dirac observable in the following way

#### Complete observables [Rovelli]

$$F_{[x,T]}(\tau) = \alpha_C^t(x) \Big|_{\alpha_C^t(T)=\tau} = x + \frac{1}{m}p_x(\tau - T)$$

- $F_{[x,T]}(\tau)$  is a Dirac observable
- $F_{[x,T]}(\tau)$  has a relational interpretation: “ $F$  gives the value of  $x$  at that point on the gauge orbit where  $T$  takes the value  $\tau$ ”

## Relational observables: General idea

### Generalization for infinite dimensional systems [\[Dierckx '04\]](#)

- infinite dimensional phase space:  $\{q_i(\sigma), p_j(\sigma')\} = \delta_{ij} \delta(\sigma, \sigma')$   
 $i, j \in$  finite index set,  $\sigma, \sigma' \in$  infinite index set (e.g. coordinates on the spatial hypersurfaces)
- first class constraint algebra  $\{C_j, C_k\} \simeq f_{ijk} C_k$
- choose one **clock function**  $T$  per constraint
- construct new set of constraints:  
 $\tilde{C}_k := (A^{-1})_K^j C_j \quad A_j^K := \{T^K, C_j\}$  (generate the same gauge orbit)
- choose phase space function  $f$

### Complete observables for field theories [\[Dierckx '04\]](#)

$$F_{[f, T^K]}(\tau^K) = \sum_0^\infty \frac{1}{n!} \{ \dots \{ f, \tilde{C}_{K_1} \}, \dots, \tilde{C}_{K_n} \} \times (\tau^{K_1} - T^{K_1}) \times \dots \times (\tau^{K_n} - T^{K_n})$$

## Relational observables: General idea

### Example: Free Maxwell theory

- phase space given by  $U(1)$ -connection  $A_a(\sigma)$  and electric fields  $E^a(\sigma)$ ,  $a = 1, 2, 3$  and  $\sigma \in \Sigma_3$

$$\{A_a(\sigma), E^b(\sigma')\} = \delta_a^b \delta(\sigma, \sigma')$$

- $U(1)$ -Gauss-constraint:  $G = \partial_a E^a(\sigma)$
- choose **clock function**  $T(\sigma) := \partial^a A_a(\sigma)$

- compute new set of constraints:  
 $\tilde{C} := [\{T, G\}]^{-1} G = -\Delta^{-1} \partial_a E^a(\sigma)$

- complete observables:

$$F_{[E^a(\sigma), T = \partial^a A_a(\sigma)]}(\tau) = E^a(\sigma)$$

$$F_{[A_a(\sigma), T = \partial^a A_a(\sigma)]}(\tau) = A_a(\sigma) - \Delta^{-1} \partial_a \partial^b A_b(\sigma)$$

- complete observables are the transversal vector modes!



## Relational observables: General idea

### Problems

- interesting result on a formal level
- but there are some problems:
  - infinite sum
  - convergence properties will strongly depend on choice of clock-variables
  - have to invert infinite dimensional matrix
$$A_j^K(\sigma, \sigma') := \{T^K(\sigma), C_j(\sigma')\}$$
- nevertheless, complete observables are an ideal starting point for perturbation theory

## A perturbative approach

### Perturbation theory around symmetry reduced sectors of the theory

- split phase space into
  - symmetry reduced sector (0th order variables)
  - anything else (higher order variables)
- can be implemented by a projection operator  $\mathcal{P}$  acting on a canonical basis
- insert split into  $F_{[f, T^K]}(\tau^K)$  and sort by powers of “fluctuations”
- “perturbative complete observable”: restrict series to the first  $(n + 1)$  terms in the series:
 
$$\{ [^n]F_{[f, T^K]}(\tau^K), C_j \} \simeq \mathcal{O}(n)$$

## A perturbative approach

### Application to homogeneous, isotropic cosmologies

- phase space of GR given by connection  $A_a(\sigma)$  and “electric” field  $E^a(\sigma)$ , Gauss-, Diffeo- and Hamiltonian constraint
- projection operator  $\mathcal{P}$  averages over dynamical degrees of freedom:

$$\mathcal{P} := \frac{1}{3} \delta_a^i \int_{\Sigma_3} \delta_i^a(\dots) \quad \beta: \text{ Immirzi parameter}$$

$$\begin{aligned} A_a^i(\sigma) &= \mathcal{P} \cdot A_a^i + (id - \mathcal{P}) \cdot A_a^i \\ &=: \underbrace{A \delta_a^i}_{\text{0th order}} + \underbrace{a_a^b(\sigma) \beta \delta_b^i}_{\text{1st order}} \end{aligned}$$

- $A$  describes the usual FRW-connection (homogeneous and isotropic),  $a_a^b(\sigma)$  inhomogeneous and anisotropic fluctuations

## Standard approach to cosmological perturbation theory

### Homogeneous and isotropic universes

- FRW-solution (for vanishing spatial curvature)  $ds^2 = a^2(\eta)(d\eta^2 - d\mathbf{x}^2)$
- put FRW-solution in Einstein's field equations and add matter  
(e.g.  $T_\nu^\mu = \text{diag}(\rho, -p, -p - p)$ )  
⇒ Friedman-equations:  $(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho$     $\dot{\rho} = -3H(\rho + p)$

### How to deal with inhomogeneities? [Brandenberger, Feldman, Mukhanov '93]

- perturbation theory:  $g_{\mu\nu} = {}^{(\text{FRW})}g_{\mu\nu} + \delta g_{\mu\nu}$
- characterize 10 entries of  $\delta g_{\mu\nu}$  according to their behaviour under  $SO(3)$ :
  - 4 scalar modes
  - 4 vector modes
  - 2 tensor modes
- What are the physical, i.e. gauge invariant degrees of freedom?

## Standard approach to cosmological perturbation theory

### Linear perturbation theory

- in linear order:
  - modes do not couple
  - can concentrate on scalar modes
  - gauge invariant modes are known: **Bardeen Potentials** [Bardeen '80]
    - invariant under *small* gauge trafo's, commute with linearized constraints
- insert Bardeen Potentials and desired matter contribution (e.g. scalar fields) into Einstein's eq. and keep only terms linear in  $\delta g_{\mu\nu}$  and  $\delta T_{\nu}^{\mu}$ 
  - ⇒ system of DE's
  - ⇒ **dynamical evolution of matter- and gravitational perturbations on a FRW-background**
- most important tool to study physics of the early universe

## Standard approach to cosmological perturbation theory

### Problems with higher order perturbation theory

- modes couple
  - problematic to extract gauge-inv. observables already to 2nd order
  - difficult to distinguish **physical perturbations** from mere **coordinate artefacts**
  - cannot trust dynamical evolution
- backreactions
  - split into **background** and **perturbations** gets ambiguous in higher order
  - background dynamics gets influenced by perturbations starting at 2nd order
    - i.e. FRW is only a 0th order approximation to the true dynamical behaviour of homogeneous and isotropic quantities
- messy calculations

# Relational approach to cosmological perturbation theory

## Outline

- based on **Relational Observables**
- “**coordinates**” are not put in by hand but defined in a **purely dynamical** way
  - use physical fields to define what **space** and **time** are
- “**background**” emerges through an averaging procedure over dynamical degrees of freedom on the spatial hypersurfaces
- symmetry reduced sector of homogeneous and isotropic fields (**background**) stays entirely dynamical
  - important for consistency in higher order perturbation theory, e.g. **backreaction effects**
- “**Feynman-like interpretation**” in terms of **free propagation** and **interactions**
  - **free propagation**: linear evolution of perturbations on FRW background
  - **interactions**: couplings between different modes

## Relational approach to cosmological perturbation theory

## projection and symplectic structure

- phase space of GR:  $\{A_a^j(\sigma), E_k^b(\sigma')\} = \kappa \delta_a^b \delta_k^j \delta(\sigma, \sigma')$ , Gauss-, Diffeo- and Hamiltonian constraint (use weight 2 version throughout this talk)
- projection operator onto homogeneous, isotropic subspace:

$$\mathcal{P} := \frac{1}{3} \delta_a^i \int_{\Sigma_3} \delta_i^a(\dots)$$

## split canonical basis:

$$\begin{aligned} A_a^i(\sigma) &= \mathcal{P} \cdot A_a^i + (id - \mathcal{P}) \cdot A_a^i \\ &=: \underbrace{A\beta\delta_a^i}_{\text{0th order}} + \underbrace{a_a^b(\sigma)\beta\delta_b^i}_{\text{1st order}} \\ E_i^a(\sigma) &= \mathcal{P} \cdot E_i^a + (id - \mathcal{P}) \cdot E_i^a \\ &=: \underbrace{E\beta^{-1}\delta_i^a}_{\text{0th order}} + \underbrace{e_b^a(\sigma)\beta^{-1}\delta_i^b}_{\text{1st order}} \end{aligned}$$

## symplectic structure

$$\begin{aligned} \{A, E\} &= \frac{\kappa}{3} \\ \{a_a^b(\sigma), e_c^d(\sigma')\} &= \\ \kappa \delta_a^c \delta_b^d \delta(\sigma, \sigma') - \frac{\kappa}{3} \delta_a^b \delta_d^c \end{aligned}$$



## Relational approach to cosmological perturbation theory

## some details

- technical details:
  - assume compact spatial topology  $\Sigma_3 = T^3$
  - work in Fourier-space
  - choose fixed value for Immirzi parameter:  $\beta = \frac{i}{2}$

## constraints for GR + scalar field

$$\text{Gauss:} \quad G_j = \partial_a E_j^a + \epsilon_{jkl} A_a^k E_l^a$$

$$\text{Diffeo:} \quad V_a = \frac{1}{\kappa} F_{ab}^j E_j^b + \frac{1}{\gamma} \pi \partial_a \phi$$

$$\text{Hamiltonian:} \quad C = \beta^2 \epsilon_{jkl} F_{ab}^j E_k^a E_l^b \\ + \frac{1}{\gamma} \left( \frac{1}{2} \pi^2 + \frac{1}{2} \beta^2 E_j^a E_j^b \partial_a \phi \partial_b \phi + \beta^3 \det(E) V(\phi) \right)$$

## Relational approach to cosmological perturbation theory

## some details

## perturbative structure of the constraints

	$k = 0$	$k \neq 0$
Gauss	$\mathcal{O}(1) + \mathcal{O}(2) + \dots$	$\mathcal{O}(1) + \mathcal{O}(2) + \dots$
Diffeo	$\mathcal{O}(2) + \mathcal{O}(3) + \dots$	$\mathcal{O}(1) + \mathcal{O}(2) + \dots$
Hamiltonian	$\mathcal{O}(0) + \mathcal{O}(2) + \dots$	$\mathcal{O}(1) + \mathcal{O}(2) + \dots$

- ${}^{(0)}C(0)$  is the FRW Hamiltonian:

$${}^{(0)}C(0) = \frac{1}{\kappa} 6\beta^2 A^2 E^2 + \frac{1}{\gamma} \left( \frac{1}{2} \pi^2 + E^3 V(\Phi) \right)$$

- integrated Diffeo-constraint starts at 2nd order

⇒ **linearization instabilities** [see e.g. Fischer, Marsden '73 or Moncrief '75/'76]

2 possibilities:

- use massless scalar fields as clocks
- one can show that one can deal with the linearization instabilities AFTER solving for the remaining constraints

## Clock variables

### 4 scalar fields as clocks

- $\varphi^0(\sigma) = \Phi^0 + \phi^0(\sigma)$        $\pi_0 = \Pi_0 + \rho_0(\sigma)$   
 $\varphi^A(\sigma) = \delta_a^A \sigma^a + \phi^A(\sigma)$        $\pi_A(\sigma) = 0 + \rho_A(\sigma)$        $A = 1, 2, 3$
- clock functions:
  - $T^{D_a}(k) = \sum_A \varphi^A(k) \delta_A^a$  (associated to the Diffeomorphism constraint)
  - $T^C(k) = \varphi^0(k)$  (associated to the Hamiltonian constraint)
  - $T^{G_a} = \epsilon^{abc} e_{bc}(k)$  (associated to the Gauss constraint)
- +**: no linearization instabilities
- : hard to compare with standard approach

## Clock variables

### clock variables out of gravitational modes

- non-local ( $\Delta^{-1}$ ) functions containing certain tensorial modes of the inhomogeneous, anisotropic parts of the gravitational fields
- e.g. **longitudinal clocks**:
  - $T^{G_a} = \epsilon^{abc} (AT e_{bc} + LT e_{bc} + TL e_{bc})$
  - $T^{D_a} = \Delta^{-1} (-\partial^a LL e_d^d + \frac{1}{2} \partial^a T e_d^d - \partial_e TL e^{ae} - \partial_d LT e^{da})$
  - $T^C = -\Delta^{-1} (\frac{1}{2} T a_d^d - LL a_d^d)$
  - plus homogeneous clock to measure “time”
- +: very close to the standard treatment
- -: non-local terms in higher orders

## Linear perturbation theory

### First order dynamics for perturbations on FRW

- calculate  ${}^{[1]}F_{[f, T^K]}(\tau^K)$  for a **first order quantity  $f$**  (e.g. scalar, vector or tensor modes of the linearized gravitational field)
- using the **longitudinal clocks** we can reproduce the standard results of linear cosmological perturbation theory, but in a purely relational way
- $\Rightarrow$  system of DE's for  ${}^{[1]}F_{[f, T^K]}(\tau^K)$   
 $\Rightarrow$  generalized propagator functions on curved backgrounds, etc. ...

## Higher order perturbation theory

## Backreaction effects

- consider complete observables for a zeroth order quantity  $f$
- $^{[0]}F_{[f, T^K]}(\tau^K) =: \alpha_{\text{free}}^{t-T^0(0)}(f)$  coincides with usual FRW dynamics for homogeneous and isotropic quantities  $f$  (e.g. scale factor, total spatial volume, ...)

- But there are higher order corrections!

- calculate  $^{[2]}F_{[f, T^K]}(\tau^K)$  for homogeneous and isotropic  $f$ :

$$\begin{aligned}
 ^{[2]}F_{[f, T^K]}(\tau^K) &= \alpha_{\text{free}}^{t-T^0(0)}(f) \\
 &\quad + \int_0^{t-T^0(0)} ds \alpha_{\text{free}}^{t-T^0(0)}[\{\alpha_{\text{free}}^s(f), {}^{(2)}\tilde{C}\}] \\
 &\quad + \text{gauge inv. extension of these terms}
 \end{aligned}$$

## An example: Bianchi-I as perturbation around FRW

### The model

- **homogeneous** but **anisotropic** cosmologies, exactly solvable

- $$A_a^j = \beta \text{diag}(A_1, A_2, A_3) \quad E_j^a = \beta^{-1} \text{diag}(E_1, E_2, E_3)$$

$$\phi(x) = \phi \quad \pi(x) = \pi$$

- Hamiltonian constraint

$$C = \frac{2\beta^2}{\kappa} (E_1 A_1 E_2 A_2 + E_1 A_1 E_3 A_3 + E_2 A_2 E_3 A_3) + \frac{1}{2\gamma} \pi^2$$

- treat Bianchi-I as perturbation around FRW

$$A := \frac{1}{3} \sum_i A_i \quad E := \sum_i E_i$$

$$a_i := A_i - A \quad e_i := E_i - E$$

### Observables

- consider complete observable associated to  $E$  and the clock  $T := \frac{\phi}{\pi}$

## An example: Bianchi-I as perturbation around FRW

## Observables

- exact solution

$$F_{[E,T]}(\tau) = \frac{1}{3} \sum_i E_i \exp[2\beta^2 A_i E_i (\tau - \frac{\phi}{\pi})] \exp[-2\beta^2 \sum_j A_j E_j (\tau - \frac{\phi}{\pi})]$$

- 0th order approximation (FRW)  $^{[0]}F_{[E,T]}(\tau) = E \exp[-\omega(\tau - \frac{\phi}{\pi})]$

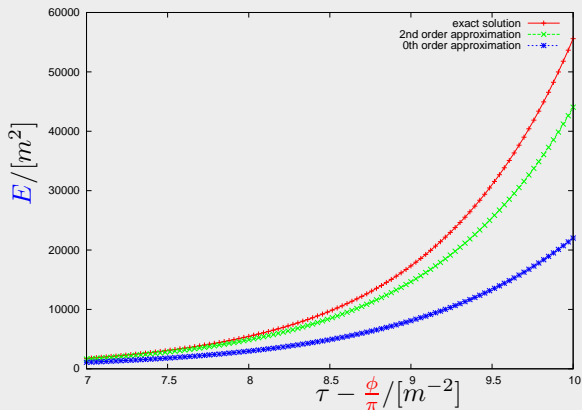
- 2nd order approximation (backreaction effects)  $\omega = 4\beta^2 A E$

$$\begin{aligned} ^{[2]}F_{[E,T]}(\tau) = & \exp[-\omega(\tau - \frac{\phi}{\pi})] \\ & (E + \frac{2}{3}\beta^2 A(\tau - \frac{\phi}{\pi}) \sum_i e_i e_i - \frac{2}{3}\beta^2 E(\tau - \frac{\phi}{\pi}) \sum_i a_i e_i \\ & + \frac{1}{6}\beta^2 \omega A(\tau - \frac{\phi}{\pi})^2 \sum_i e_i e_i + \frac{1}{6}\beta^2 \omega \frac{E^2}{A}(\tau - \\ & \frac{\phi}{\pi})^2 \sum_i a_i a_i + \frac{1}{3}\beta^2 \omega E(\tau - \frac{\phi}{\pi})^2 \sum_i a_i e_i) \end{aligned}$$



## An example: Bianchi-I as perturbation around FRW

## Observables



initial conditions:  $A = 1$ ,  $E = 1m^2$ ,  $a_1 = a_2 = 0.1$ ,  $a_3 = -0.2$ ,  $e_1 = e_2 = 0.1m^2$ ,  $e_3 = -0.2m^2$

## An example: Bianchi-I as perturbation around FRW

### What we learned from Bianchi-I:

- calculation of **backreaction effects** using relational observables is straight forward
- **FRW is not a good approximation** to the true dynamical behaviour of homogeneous and isotropic quantities for all times
- **backreaction effects are physical effects** (gauge invariance), not mere coordinate artefacts
- model is too simple to draw any conclusion for the “real world”
- $\Rightarrow$  use relational observables to calculate backreactions in a physically more realistic model
- for which range of initial conditions does perturbation theory make sense?

## Outlook and Conclusions

- Relational observables are well suited to describe dynamics in GR
- Applying the relational framework to cosmology has several advantages compared to the standard approach:
  - issue of gauge invariance is very clear
  - one scheme for arbitrary orders
  - nice interpretation: “background” emerges through averaging over dynamical degrees of freedom
- ideal framework to calculate backreaction effects in cosmology
- can backreaction effects possibly resolve the dark energy problem? [see e.g. Buchert [gr-qc/0612166](#)]