Bubble divergences in state-sum models

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Introduction: the Ponzano-Regge state-sum model

The Ponzano-Regge state-sum model is formally defined by

$$\mathcal{Z}_{\mathrm{PR}}(\Delta_2^*) = \sum_{\{j_f\}} \prod_f (2j_f + 1) \prod_v \{6j\}$$

where Δ_2^* is the dual 2-skeleton of a triangulated 3-manifold Δ.

[Ponzano, Regge (68)]

This expression is almost always divergent. Understanding the structure of these divergences is crucial for

- Spinfoam models, of which the PR model is the epitome.
- Group field theory, where they might generate a renormalization flow.
- Quantum topology, in order to define a *Ponzano-Regge invariant*.

Outline

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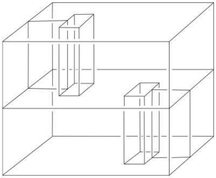
Counting the vertices of the triangulation?

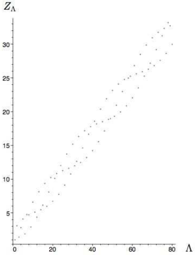
Ponzano and Regge associated these divergences to the vertices of the simplicial complex Δ, and proposed the improved definition

$$\mathcal{Z}'_{\mathrm{PR}}(\Delta_2^*) = \lim_{\Lambda \to \infty} \frac{1}{\Lambda^{3|\Delta_0|}} \sum_{\{j_f\}}^{\Lambda} \prod_f (2j_f + 1) \prod_v \{6j\}.$$

[Ponzano, Regge (68)]

Unfortunately, this fails.





[Barrett, Naish-Guzman (09)]

Flat connections and discrete Bianchi identity

The Ponzano-Regge model can be given a gauge-theoretic definition, as the partition function of a system of flat connections.

$$\mathcal{Z}_{\mathrm{PR}}(\Delta_2^*) = \int_{\mathrm{SU}(2)^E} dA \prod_f \delta\big(H_f(A)\big)$$

- Discrete connection: $A = (g_e)_e \in \mathrm{SU}(2)^E$
- Haar measure: $dA = \prod_e dg_e$

In this setting,

- Curvature of A:

$$H(A) = (H_f(A) = \prod_{e \in \partial f} g_e^{\pm 1})_f \in \mathrm{SU}(2)^F$$

- Gauge transformation of A along $k \in \mathrm{SU}(2)$ (assume single vertex):

$$\gamma_A(h) = (k g_e k^{-1})_e.$$

Freidel and Louapre's proposal

Some of these δ functions are redundant, as there are discrete Bianchi identities: for each vertex $v \in \Delta_0$, there is an ordering of the faces surrounding it such that

$$\overrightarrow{\prod_f} H_f^{\pm 1} = 1.$$

Freidel and Louapre then proposed to collapse a spanning tree in Δ to remove these redundancies. This amounts to removing a tree of faces in Δ_2^*, yielding

$$\mathcal{Z}'_{FL}(\Delta_2^*) = \int_{\mathrm{SU}(2)^E} dA \prod_{f \in \Delta_2^* \setminus T} \delta\big(H_f(A)\big)$$

[Freidel, Louapre (03)]

Counter-examples

For lens spaces, there are triangulations such that Δ_2^* has only one face, and

$$\mathcal{Z}'_{FL}(\Delta_2^*) = \int_{\mathrm{SU}(2)} dg\ \delta(g^p) = \infty.$$

The same happens for the 3-torus.

"In general we do not expect this invariant to be finite for topologically non trivial closed manifold."

[Freidel, Louapre (03)]

Counting the bubbles of the foam?

It was proposed that these are higher analogues of loop divergences, arising because of the spins get unbounded along bubbles: collections of faces forming closed surfaces.

[Perez, Rovelli (00)]

In 3 dimensions, there is correspondance between vertices of Δ and bubbles of Δ_2^*. This correspondance breaks down in four dimensions. The notion of bubble divergence is the more general one.

This idea was recently used to estimate the divergence degree for certain foams, coined 'type 1':

$$\mathcal{Z}_{\mathrm{PR}}(\text{type 1}) = \left(\sum_{j=0}^{\Lambda} (2j+1)^2 \right)^{B-1} .$$

[Freidel, Gurau, Oriti (09)]

Our goal: divergence degree and dominant part

We consider the regularized expression

$$\mathcal{Z}_\tau(\Gamma, G) = \int_{\mathrm{SU}(2)^E} dA \prod_f K_\tau\Big(H_f(A)\Big)$$

with

- Γ an arbitrary cell 2-complex (manifold or not) with one vertex
- G a compact (semi-simple) Lie group
- K_τ the heat kernel on G, $K_\tau(g) \underset{\tau \to 0}{\sim} \Big(\underbrace{\frac{1}{\sqrt{4\pi\tau}}}_{\Lambda_\tau}\Big)^{\dim G} \exp\Big(-\frac{|g|^2}{4\tau}\Big)$

[Freidel, Louapre (03)]

and look for an asymptotic estimate of the form

$$\mathcal{Z}_\tau(\Gamma, G) \underset{\tau \to 0}{\sim} \Lambda_\tau^{\Omega(\Gamma, G)} \underbrace{\mathcal{Z}'(\Gamma, G)}_{<\infty}$$

An implicit assumption

In previous investigations, it was always implicitely assumed that the divergences can be captured by a purely combinatorial criterion:

- vertices in Δ (Ponzano-Regge, Freidel-Louapre, Barrett-Naish-Guzman)
- bubbles in Δ_2^* (Perez-Rovelli, Freidel-Gurau-Oriti)

This implies that Ω is a multiple of $\dim G$.

This is not true in general.

This is why the Ponzano-Regge, or Freidel-Louapre, regularizations fail, and why the Freidel-Gurau-Oriti estimate cannot be general.

Our results

- The combinatorial powercounting is true in trivial cases
  - Γ simply connected
  - G Abelian

  where indeed
  $$\Omega(\Gamma, G) = (\dim G) b_2(\Gamma).$$

- In more general cases, this formula is twisted, and Ω is not a multiple of $\dim G$:
  $$\Omega(\Gamma, G) = \widetilde{b_2}.$$

- (The dominant part $\mathcal{Z}'(\Gamma, G)$ can be related to Reidemeister torsion, work in progress.)

Outline

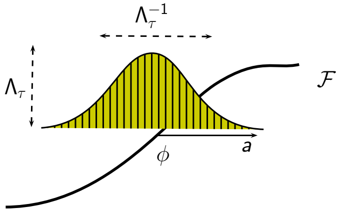
Generalized Laplace approximation

The integral

$$\mathcal{Z}_\tau(\Gamma, G) = \int_{\mathrm{SU}(2)^E} dA \prod_f K_\tau\Big(H_f(A)\Big) \underset{\tau\to 0}{\sim} \Lambda_\tau^{(\dim G)F} \int_{\mathrm{SU}(2)^E} dA\, e^{\frac{-\sum_f |H_f(A)|^2}{4\tau}}$$

is peaked on the set $\mathcal{F}$ of flat connections ϕ, for which $H(\phi) = 1$. In the neighborhood of $\mathcal{F}$, we have $A = \exp_\phi(a)$ for $a \in N_\phi\mathcal{F}$, and

$$\sum_f |H_f(A)|^2 = \|dH_\phi(a)\|^2_{\mathfrak{g}^F}.$$



Gaussian integrals transversally to $\mathcal{F}$.

[Forman (93)]

A caveat: singular connections.

However, because $1 \in G^F$ is usually not a regular value of the smooth map H, i.e. H is not submersive on $\mathcal{F}$, $\mathcal{F}$ is not a manifold, but rather an 'algebraic set'.

The singularities of $\mathcal{F}$ are the connections ϕ such that

$$\ker dH_\phi \neq T_\phi \mathcal{F}.$$

We assume they do not contribute to the integral.

- True in two dimensions. [Sengupta (03)]
- We know one counter-example, see our paper.

The non-singular flat connection do form a manifold. Since

$$\dim \ker dH_\phi \geq \dim T_\phi \mathcal{F},$$

they are the flat connections where H has maximal rank.

Powercounting

The Gaussian integrals bring about convergent factors, one per transverse direction:

$$\int_{N_\phi \mathcal{F}} da\ e^{-\|dH_\phi(a)\|^2_{\mathfrak{g}^F}/4\tau} = \Lambda_\tau^{-\dim N_\phi \mathcal{F}} \quad \underbrace{\det\left((dH_\phi^\perp)^\dagger dH_\phi^\perp\right)^{-1/2}}_{\text{Gaussian determinant, indep. of } \tau}$$

Hence

$$\mathcal{Z}_\tau(\Gamma, G) = \Lambda_\tau^{\Omega(\Gamma,G)} \int_{\mathcal{F}} d\phi\ f(\phi),$$

with

$$\Omega(\Gamma, G) = (\dim G)F - \dim N_\phi \mathcal{F}$$

i.e.

$$\Omega(\Gamma, G) = (\dim G)F - \max_{\mathcal{F}} \mathrm{rk}\ H.$$

Cohomological interpretation

Our result can be given a cohomological interpretation. This is a neat way to disentangle, about a flat connection ϕ, the variations $a \in T_\phi G^E$ which

- leave ϕ flat ($a \in \ker dH_\phi$)
  - because they are infinitesimal gauge transformations ($a \in \mathrm{Im}\ d\gamma_\phi$)
  - not for this reason ($a \notin \mathrm{Im}\ d\gamma_\phi$)
- introduce curvature ($a \notin \ker dH_\phi$)

$$\underbrace{C^0_\phi = \mathfrak{g}}_{\text{inf. gauge transfo.}} \xrightarrow{d\gamma_\phi} \underbrace{C^1_\phi = T_\phi G^E}_{\text{variations about } \phi} \xrightarrow{dH_\phi} \underbrace{C^2_\phi = \mathfrak{g}^F}_{\text{inf. holonomies}}$$

[Witten (89), Barrett-Naish-Guzman (09)]

$\Omega(\Gamma, G) = b^2_\phi$ is the second Betti number in this twisted cohomology.

Note: when $\phi = 1$, this is nothing but the cellular cohomology of Γ with coefficients in $\mathfrak{g}$, and then

$$\Omega(\Gamma, G) = (\dim G) b^2(\Gamma).$$

Outline

Triangulation dependence of the divergence degree

Assume now that Γ is the dual 2-skeleton of a triangulation $\Delta^{(d)}$ of a d-manifold $M^{(d)}$.

Elementary manipulations on the expression of $\Omega(\Gamma, G)$ yield

$$\Omega(\Delta^{(d)}, G) = \underbrace{\dim \mathcal{M} - \dim \zeta + \Big(\dim G\Big)\chi(M^{(d)})}_{\text{topological invariant}} \\ \underbrace{+ \Big(\dim G\Big) \sum_{j=0}^{d-3} (-1)^{d+j} |\Delta_j^{(d)}|}_{\text{triangulation dependent}}.$$

with

- $\mathcal{M}$ is the moduli space of flat connections
- ζ is the isotropy group of non-singular flat connections
- $|\Delta_j^{(d)}|$ the number of j-simplices

Three dimensions

In three dimensions, this becomes

$$\Omega(\Delta^{(d)}, G) = \dim \mathcal{M} - \dim \zeta \\ +\big(\dim G\big)|\Delta_0^{(3)}|.$$

Back to Ponzano and Regge's original intuition ("divergences are associated to vertices of the triangulation"):

- They missed the topological term, and this is why their regularization failed.
- But! They were right about the variation of Ω in a Pachner move:

$$\delta_{\mathrm{Pachner}}\left(\Omega(\Delta^{(d)}, G)\right) = (\dim G)\delta_{\mathrm{Pachner}}\left(|\Delta_0^{(3)}|\right).$$

Four dimensions

In four dimensions, the formula becomes

$$\Omega(\Delta^{(d)}, G) = \dim \mathcal{M} - \dim \zeta + (\dim G)\chi(M^{(4)}) + \left(\dim G\right)\left(|\Delta_1^{(4)}| - |\Delta_0^{(4)}|\right).$$

Again, the variation of Ω in a Pachner move is correctly captured by the combinatorial estimate, the number of bubbles being

edges - vertices.

Conclusions

- The divergence degree of a foam is given by the nomber of transverse directions to the set of flat connections.
- The notion that it counts the "number of bubbles" is correct, but in a subtle sense: Ω is the second Betti number in a twisted cohomology. In particular it is not a multiple of $\dim G$.
- In the case of manifolds, the old arguments relying on Pachner moves capture the variation of Ω, but not Ω itself.

Can these methods be used to study the gravitational models? We do not know.

Thanks!