Stepping out of Homogeneity in Loop Quantum Cosmology

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Outline

1. Why inhomogeneities?
2. Cosmological principle
3. The models: classical theory
4. The models: quantum theory
5. n=2 case: dipole cosmology
6. Born-Oppenheimer approximation
7. Recovering LQC
8. Open questions

Questions

LQC provides the most successful physical application of loop gravity, and one of the most promising avenues towards a possible empirical test.

1. Can we include inhomogeneities? Inhomogeneities can be re-inserted at a later cosmological epoch, restricting the analysis of the Planck epoch to the sole homogeneous d.o.f.. Is this approximation sufficient? Do quantum fluctuations of the inhomogeneous d.o.f. play any role at the bounce? (See also [Martin-Benito, Garay, Mena Marugan]).
2. Can we describe the full quantum geometry at the bounce, beyond the homogeneous approximation?
3. Quantum fluctuations of the inhomogeneous d.o.f. are believed to play a major role in structure formation. Can we see them in our background independent language?
4. Relation LQC and LQG. Can we derive the LQC assumptions from a more complete LQG model?

A strategy to address these questions

1. Analyze the nature of the approximation on which cosmology –classical or quantum– is based.

2. This leads to the idea that the full theory may be expanded by adding degrees of freedom one by one, starting from the cosmological ones. Define an approximated dynamics of the universe, inhomogeneous but truncated at a finite number of d.o.f..

3. Roughly a mode expansion, or an approximation of the universe by a triangulation with n tertrahedra.

4. At fixed n, approximate the dynamics by the non-graph changing Hamiltonian constraint. This gives a consistent classical and quantum model for each n.

5. Study the simple $n = 2$ case ("dipole cosmology"). This provides a "first step out of inhomogeneity".

6. Within this model, single out the homogeneous d.o.f., and perform a Born-Oppenheimer approximation. Recover the structure of LQC.

Approximations in cosmology

1. Einstein's 1917: *cosmological principle*. The dynamics of a homogeneous and isotropic space describes our real universe. What does this mean?

2. The universe happens to be in a state where the effect of the inhomogeneities on the dynamics of its largest scale, described by the scale factor, can be neglected in a first approximation.

3. This is not a large scale approximation, because it is supposed to remain valid when the universe was small! It is an expansion in $n \sim \frac{a}{\lambda}$.

4. Restrict the dynamics to a finite n.

5. The large scale d.o.f. can be captured by averaging the metric over the simplices of a triangulation formed by n simplices.

The model: definition of the classical theory

- Oriented triangulation Δ_n of a 3-sphere, with by n tetrahedra t and $2n$ triangles f.

- Variables
$$\begin{cases} U_f & \in SU(2), \quad (1) \\ E_f = E_f^i \tau_i & \in su(2). \quad (2) \end{cases}$$

($U_{f^{-1}} = U_f^{-1}$ and $E_{f^{-1}} = -U_f E_f U_f^{-1}$.)

- Poisson brackets
$$\begin{cases} \{U_f, U_{f'}\} = 0, & (3) \\ \{E_f^i, U_{f'}\} = \delta_{ff'}\ \tau^i U_f, & (4) \\ \{E_f^i, E_{f'}^j\} = \delta_{ff'}\ \epsilon^{ijk} E_f^k. & (5) \end{cases}$$

The phase space is the cotangent bundle of $SU(2)^{2n}$ with its natural symplectic structure.

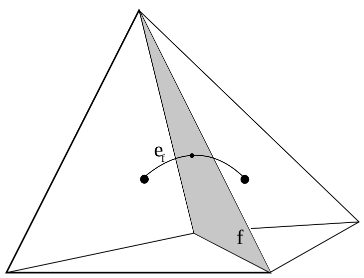
- Dynamics
$$\begin{cases} \text{Gauge} \quad G_t \equiv \sum_{f\in t} E_f \sim 0, & (6) \\ \text{Ham} \quad C_t \equiv \sum_{ff'\in t} Tr[(U_{ff'} - U_{f'f}) E_f E_{f'}] \sim 0. & (7) \end{cases}$$

where $U_{ff'} = U_f U_n \ldots U_1 U_{f'}$ is the holonomy of the link of the edge where f and f' meet.

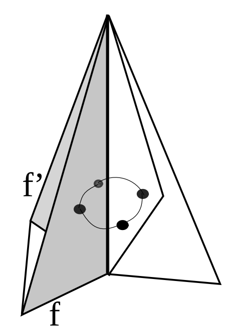
The model: interpretation

- Cosmological approximation to the dynamics of the geometry of a closed universe.
- (U_f, E_f) average gravitational d.o.f. over a triangulation Δ_n of space:

- U_f: parallel transport of the Ashtekar connection A_a along the link e_f of Δ_n^* dual to the f;
- E_f: flux Φ_f of the Ashtekar's electric field E^a across the triangle f, parallel transported to the center of the tetrahedron: $E_f = U_{e_1}^{-1} \Phi_f U_{e_1}$.



- The constraints approximate the Ashtekar's gauge and Hamiltonian constraint $\mathrm{Tr}[F_{ab} E^a E^b] \sim 0$.
- C_t: "Non-graph-changing" hamiltonian constraint.
- $U_{ff'} \sim \mathbb{1} + |\alpha|^2 F_{ab} + o(|\alpha|^2 F)$ is also good for large loops if F_{ab} is small. $\longrightarrow$ Approximation does not get worse as the universe increases its size. (Near flat spacetime the approximation is good even for coarse triangulations!)



The model: adding a scalar field

- Add a variable (ϕ_t, p_{ϕ_t}). Represents matter, defines an n-fingered time.
- Hamiltonian constraint

$$S_t = \frac{1}{V_t} C_t + \frac{\kappa}{2V_t}\, p^2_{\phi_t} \sim 0.$$

  where

$$V_t = \sum_{f\,f'\,f''\in t} \sqrt{Tr[E_f E_{f'} E_{f''}]}.$$

- Ultralocal.
- Easy to add spatial derivative terms, or extend to fermions and gauge fields.

The model: quantum theory

1. Hilbert space: $H_{aux} = L_2[SU(2)^{2n}, dU_f]$. States $\psi(U_f)$.

2. Operators: U_f are diagonal and E_f are the left invariant vector fields on each $SU(2)$. The operators $E_{f^{-1}}$ turn then out to be the right invariant vector fields !

3. States that solve gauge constraint: $SU(2)$ spin networks on graph Δ_n^*

$$\psi_{j_f \iota_t}(U_f) \equiv \langle U_f | j_f, \iota_t \rangle \equiv \otimes_f \ \Pi^{(j_f)}(U_f) \cdot \otimes_t \ \iota_t. \tag{1}$$

4. With a scalar field: $H_{aux} = L_2[SU(2)^{2n}, dU_f] \otimes L_2[R^n]$, with

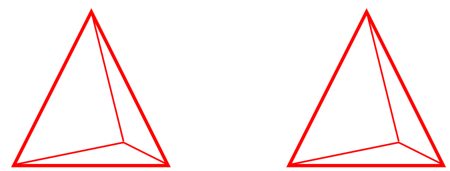
$$\psi(j_f, \iota_t, \phi_t) \equiv \langle j_f, \iota_t, \phi_t | \psi \rangle. \tag{2}$$

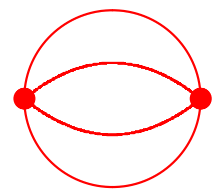
5. Quantum Hamiltonian constraint: either or as it is, or *à la* Thiemann, rewriting it in the Thiemann's form

$$\frac{1}{V_t} C_t = \frac{1}{6} \sum_{f f' f'' \in t} \mathrm{Tr} \left[(U_{ff'} - U_{f'f}) U_{f''}^{-1} [U_{f''}, V_t] \right] \sim 0. \tag{3}$$

Dipole cosmology

Take $n = 2$ and Δ_2 formed by two tetrahedra glued along all their faces.

$\Delta_2 =$  (1)

$\Delta_2^* =$  (2)

$\mathcal{H}_{aux} = L_2[SU(2)^4] \otimes L_2[R^2]$. Gauge invariant states $\psi(j_f, \iota_t, \phi_t)$.
Spin networks basis $|j_f, \iota_t, \phi_t\rangle = |j_1, j_2, j_3, j_4, \iota_1, \iota_2, \phi_1, \phi_2\rangle$.

Dynamics:

$$\left\{ \begin{array}{lcl} \dfrac{\partial^2}{\partial \phi_1^2} \psi(j_f, \iota_t, \phi_t) & = & \dfrac{2}{\kappa} \displaystyle\sum_{\epsilon_f = 0, \pm 1} {C_1}_{j_f \, \iota_t}^{\epsilon_f \, \iota'_t} \; \psi\left(j_f + \dfrac{\epsilon_j}{2}, \iota'_t, \phi_t\right), \\ \dfrac{\partial^2}{\partial \phi_2^2} \psi(j_f, \iota_t, \phi_t) & = & \dfrac{2}{\kappa} \displaystyle\sum_{\epsilon_f = 0, \pm 1} {C_2}_{j_f \, \iota_t}^{\epsilon_f \, \iota'_t} \; \psi\left(j_f + \dfrac{\epsilon_j}{2}, \iota'_t, \phi_t\right). \end{array} \right. \tag{3}$$

Born-Oppenheimer approximation

B & O have opened the way to the quantum theory of molecules.

1. D.o.f.: "heavy" (R, p_R) (nuclei) and "light" (r, p_r) (electrons).
2. Hamiltonian splits as $H(R, r, p_R, p_r) = H_R(R, p_R) + H_r(R; r, p_r)$

   H_R= nuclei energy; H_r= electrons energy plus interaction energy.
3. B-O Ansatz: $\psi(R, r) = \Psi(R)\phi(R; r)$, where $\partial_R \Phi(R; r)$ is small.
4. Time independent Schrödinger equation $H\psi = E\psi$ becomes $H\psi = (H_R + H_r)\Psi\Phi = \Phi H_R \Psi + \Psi H_r \Phi = E\Psi\Phi$ that is $\frac{H_R \Psi}{\Psi} - E = -\frac{H_r \Phi}{\Phi}$. Since the lhs does not depend on r, each side is equal to a function $\rho(R)$. Therefore we can write two equations

$$\begin{cases} H_R \Psi(R) + \rho(R)\Psi(R) = E\Psi(R). & (1) \quad \text{Schr. eq. for nuclei, with additional term.} \\ H_r \Phi(R, r) = \rho(R)\Phi(R, r). & (2) \quad \text{Schr. eq. for electrons, in the background } R. \end{cases}$$

Let us apply this idea to the dipole cosmology: $R \to$ hom d.o.f., $r \to$ inhom d.o.f..

Homogeneous and inhomogeneous d. o. f.

1. $U_f = expA_f$. ω_f: fixed fiducial background. ($|\omega_f| = 1$).

2. Degrees of freedom

$$\begin{cases} A_f = c\,\omega_f + a_f\,, & (1) \\ E_f = p\,\omega_f + h_f\,. & (2) \end{cases} \quad \text{where} \quad \begin{cases} V = p^{\frac{3}{2}}\,, & (3) \\ \{c,p\} = \frac{8\pi G}{3} \equiv k. & (4) \end{cases}$$

Also $\phi_{1,2} = \frac{1}{2}(\phi \pm \Delta\phi)$, and $V_{1,2} = \frac{1}{2}(V \pm \Delta V)$.

3. Hamiltonian constraint: $C_t = C_t^{hom} + C_t^{in}$. Where: C_t^{hom}: homogeneous d.o.f. energy. C_t^{in} sum of inhomogeneous d.o.f. and interaction energy.

$$C_t(c,a,p,h) = \frac{1}{12}\sum_{ff'f''\in t} \mathrm{Tr}\Big[\Big(e^{-c\omega_{f'}-a_{f'}}\,e^{c\omega_f - a_f} - e^{-c\omega_f - a_f}\,e^{c\omega_{f'}-a_{f'}}\Big) e^{-c\omega_{f'f''}-a_{f'f''}}\,[e^{c\omega_{f'f''}+a_{f'f''}}, V \pm \Delta V]\Big]. \quad (5)$$

$$C_t^{hom}(c,p) = \frac{1}{12}\sum_{ff'f''\in t} \mathrm{Tr}\Big[\Big(e^{-c\omega_{f'}}\,e^{c\omega_f} - e^{-c\omega_f}\,e^{c\omega_{f'}}\Big)\,e^{-c\omega_{f''}}\,[e^{c\omega_{f''}}, p^{\frac{3}{2}}]\Big] \quad (6)$$

4. B-O Ansatz: $\psi(c,a,\phi,\Delta\phi) = \Psi(c,\phi)\phi(c,\phi;a\Delta\phi)$.

5. With some technicalities:

$$\begin{cases} \dfrac{\partial^2}{\partial\phi^2}\Psi(c,\phi) - C^{hom}\Psi(c,\phi) - \rho(c,\phi)\Psi(c,\phi) = 0, & (1) \\ \dfrac{\partial^2}{\partial\phi^2}\phi(c,\phi;a,\Delta\phi) + C^{inh}\phi(c,\phi;a,\Delta\phi) = \rho(c,\phi)\phi(c,\phi;a,\Delta\phi). & (2) \end{cases}$$

(1): Quantum Friedmann equation for the homogeneous d.o.f. (c,ϕ), corrected by the energy density $\rho(c,\phi)$ of the inhomogeneous modes.

(2): The Schrödinger equation for the inhomogeneous modes in the background homogeneous cosmology (c,ϕ). $\rho(c,\phi)$ energy eigenvalue.

6. At the order zero of the approximation, where we disregard entirely the effect of the inhomogeneous modes on the homogeneous modes, we obtain

$$\frac{\partial^2}{\partial\phi^2}\Psi(c,\phi) = C^{hom}\Psi(c,\phi). \tag{3}$$

Let now now write this equation explicitly.

Quantum Friedmann equation

- c multiplies the generator of a $U(1)$ subgroup of $SU(2)^4$. Therefore it is a periodic variable $c \in [0, 4\pi]$. We can therefore expand $\Psi(c, \phi)$

$$\Psi(c,\phi) \quad = \sum_{\text{integer } \mu} \psi(\mu,\phi)\; e^{i\mu c/2}. \tag{1}$$

The basis of states $\langle c\,|\mu\rangle = e^{i\mu c/2}$ satisfies

$$p|\mu\rangle \quad = \quad \frac{k}{2}|\mu\rangle \tag{2}$$

$$e^{ic}|\mu\rangle \quad = \quad |\mu + 2\rangle \tag{3}$$

- The homogeneous hamiltonian constraint operator reads

$$\begin{aligned} C_t^{hom} \; &= \frac{1}{12}\sum_{ff'f''} Tr\Big[((\cos\frac{c}{2}\mathbb{1} - 2\sin\frac{c}{2}\omega_{f'})(\cos\frac{c}{2}\mathbb{1} + 2\sin\frac{c}{2}\omega_f) \\ &\qquad - \quad (\cos\frac{c}{2}\mathbb{1} - 2\sin\frac{c}{2}\omega_f)(\cos\frac{c}{2}\mathbb{1} + 2\sin\frac{c}{2}\omega_{f'}))e^{-c\omega_{f''}}[e^{c\omega_{f''}}, p^{\frac{3}{2}}]\Big] \\ &= \frac{1}{12}\sum_{ff'f''} \mathrm{Tr}\Big[\Big(2\sin\frac{c}{2}\cos\frac{c}{2}\;(\omega_f - \omega_{f'}) - 4\sin^2\frac{c}{2}\;[\omega_f, \omega_{f'}]\Big)\; e^{-c\omega_{f''}}[e^{c\omega_{f''}}, p^{\frac{3}{2}}]\Big]\,. \end{aligned}$$

- Consider the action of the last factor on the state $|\mu\rangle$

$$
\begin{aligned}
e^{-c\omega_{f''}}[e^{c\omega_{f''}}, p^{\frac{3}{2}}]e^{i\mu c/2} &= \left(-ik\frac{\partial}{\partial c}\right)^{\frac{3}{2}} e^{i\mu c/2} - e^{-c\omega_{f''}}\left(-ik\frac{\partial}{\partial c}\right)^{\frac{3}{2}} e^{ic(\mu/2 - i\omega_{f''})} \\
&= k\left(\mu^{\frac{3}{2}}\mathbb{1} - \left(\mu\mathbb{1} - i2\omega_{f''}\right)^{\frac{3}{2}}\right) e^{i\mu c/2}.
\end{aligned} \tag{1}
$$

- We can write $\left(\mu\mathbb{1} - i2\omega_{f''}\right)^{\frac{3}{2}} = \alpha(\mu)\mathbb{1} + \beta(\mu)\omega_{f''}$ where the coefficients $\alpha(\mu)$ and $\beta(\mu)$ can be easily computed squaring this equation, using $\omega_{f''}^2 = -\frac{1}{4}\mathbb{1}$ and solving the resulting system, which gives

$$
\beta(\mu) = -\sqrt{-2\mu(\mu^2+3) + 2(\mu^2-1)^{\frac{3}{2}}}\ . \tag{2}
$$

- The only term that survives is

$$
\begin{aligned}
C_t^{hom} e^{i\mu c/2} &= -\frac{1}{3}\sin^2\frac{c}{2}\beta(\mu)\sum_{ff'}\mathrm{Tr}\left[[\omega_f, \omega_{f'}]\omega_{f''}\right] e^{i\mu c/2} \\
&= C\,\beta(\mu)\,\sin^2\frac{c}{2}\ e^{i\mu c/2}.
\end{aligned} \tag{3}
$$

- Bringing everything together, the quantum Friedmann equation reads

$$\boxed{\frac{\partial^2}{\partial\phi^2}\Psi(\mu,\phi) = C^+(\mu)\ \Psi(\mu+2,\phi) + C^0(\mu)\ \Psi(\mu,\phi) + C^-(\mu)\ \Psi(\mu-2,\phi)}$$

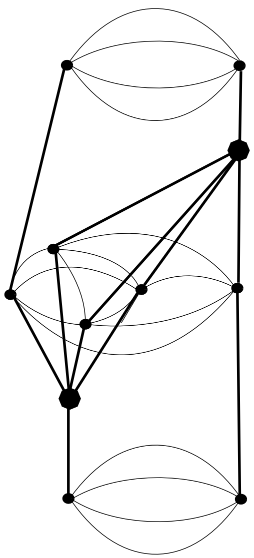
where

$$C^+(\mu) = C^-(\mu) = -\frac{1}{2}C^0(\mu) = -\frac{kC}{\kappa}\mu^{\frac{3}{2}}\sqrt{-2\mu(\mu^2+3)+2(\mu^2-1)^{\frac{3}{2}}}.$$

- This eq. has precisely the structure of the LQC dynamical equation.
- μ is discrete without ad hoc hypotheses, or area-gap argument.

To be done

1. Immirzi parameter γ.
2. Realistic matter fields.
3. Relation between the $\psi(\mu, \phi)$ homogeneous states and the full $\psi(j_f, \iota_t, \phi_t)$ states in the spinnetwork basis.
4. Relation to $\bar{\mu}$ quantization scheme.
5. Spinfoam version. Cosmological Regge calculus (Barrett et al). $1 \rightarrow 4, 4 \rightarrow 1$ Pachner moves.



Summary

1. Family of models opening a systematic way for describing the inhomogeneous d.o.f. in quantum cosmology. Does bounce scenario survives? $\psi(j_f, i_t, \phi_{bounce})$: fluctuating geometry at the bounce.
2. Derivation of the structure of LQC as a Born-Oppenheimer apprimation. Light on LQC/LQG relation.

Comments

1. Quantum inhomog. fluctuations. Role in structure formation? Inflation?
2. $\rho(c, \phi)$ term in quantum Friedmann eq. Physics? Cosmological constant?
3. Coarse triangulations. Intuition that near-flat-space dynamics can *only* be described by many nodes is misleading. If this was the case, cosmology would be ill-conceived. Relevant for the n-point functions calculations.

(ps.: Francesca is applying for Ph.D. !)