Casimir effect on a quantum geometry

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Motivation

1) The quantization of spherically symmetric vacuum spacetimes in loop quantum gravity is better understood,
   i) singularity resolution,
   ii) effective discrete geometry (quantization of the areas of symmetry).
2) What this quantum model has to do about Hawking radiation, black hole evaporation, etc.?
3) The stress-energy tensor plays a key role: codifies the density and flux of energy of test fields.
4) It is naturally regularized due to the discreteness.
5) In which manner can this affect to the traditional QFT predictions like the Casimir effect?

Plan of the talk

1) Brief review about the polarization of the vacuum for a test scalar field on Schwarszchild spacetimes.
2) Quantization of spherically symmetric vacuum spacetimes.
   i) Effective space-times.
   ii) Test fields and vacuum polarization.
3) Casimir effect.
   i) Standard computation on QFT.
   ii) Genuine calculation on the effective (discrete) loop geometry.

Test fields on Schwarszchild spacetimes: traditional analysis

Prior to the computation of the Casimir effect, it is enlightening to discuss the calculation of the polarization, as significant differences arise w.r.t. QFT in CST (Candelas, 1980).

- Let us start with a massless scalar field ϕ on a Schwarszchild spacetime.

$$\phi = \sum_{\ell m} \int_0^\infty d\omega \frac{a_{\ell,m}(\omega)}{\sqrt{4\pi\omega} r} e^{-i\omega t} Y_{\ell,m}(\theta,\phi) R_\ell(\omega|r) + \text{c.c.}, \quad (1)$$

such that $\dfrac{d^2R}{dr_*^2} + \omega^2 R - \left(1 - \dfrac{r_S}{r}\right)\left[\dfrac{\ell(\ell+1)}{r^2} + \dfrac{r_S}{r^3}\right] R = 0$, and

where $r_* = r + r_S \log(r/r_S - 1)$ and $r_S = 2M$. There are two independent solutions $\overrightarrow{R_\ell}(\omega|r)$ and $\overleftarrow{R_\ell}(\omega|r)$ fulfilling boundary conditions compatible with the Boulware vacuum.

Test fields on Schwarszchild spacetimes: traditional analysis

- The polarization of the vacuum $\langle \phi(x)^2 \rangle = \infty$.

- The 2-point $G(x, x') = \langle \phi(x)\phi(x') \rangle$ diverges if $x \to x'$ with the inverse of the geodesic distance σ.

- The renormalized vacuum polarization on the Boulware vacuum is

$$\langle \phi(x)^2 \rangle_{\text{ren}} = \lim_{\epsilon \to 0} \int_0^\infty d\omega \frac{e^{-i\omega\epsilon}}{16\pi^2\omega} \sum_\ell (2\ell + 1) \left[|\overrightarrow{R_\ell}(\omega|r)|^2 + |\overleftarrow{R_\ell}(\omega|r)|^2 \right]$$
$$- \frac{1}{4\pi^2(1 - r_S/r)\epsilon^2} - \frac{M^2}{48\pi^2 r^4 (1 - r_S/r)}. \quad (2)$$

It diverges on the horizon (i.e. for $r \to r_S$).

Loop quantization of a spherically symmetric spacetime: constraint algebra

- A complete quantization within the Dirac approach has been recently obtained (R. Gambini, J. Pullin, 2013) thanks to a suitable modification of the constraint algebra $(H, H_r) \to (\tilde{H}, H_r)$:

$$\tilde{H} := \frac{(E^x)'}{E^\varphi} H - 2\frac{\sqrt{E^x}}{E^\varphi} K_\varphi H_r = \left[\sqrt{E^x}\left(1 - \frac{[(E^x)']^2}{4(E^\varphi)^2} + K_\varphi^2\right)\right]', \qquad (3)$$

with (E^x, E^φ) and (K_x, K_φ) the triad and connection (curvature) components, respectively.

- The new constraint algebra is

$$\{H_r(N_r), H_r(\tilde{N}_r)\} = H_r(N_r\tilde{N}_r' - N_r'\tilde{N}_r),\ \{\tilde{H}(N), H_r(N_r)\} = \tilde{H}(N_r N'),$$
$$\{\tilde{H}(N), \tilde{H}(\tilde{N})\} = 0.$$

- Suitable boundary conditions for the mass M.

Loop quantization of a spherically symmetric spacetime: physical states

- Kinematical states are given by the spin networks

$$T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi) = \prod_{e_j \in g} \exp\left(\frac{i}{2} k_j \int_{e_j} dx\, K_x(x)\right) \prod_{v_j \in g} \exp\left(\frac{i}{2} \mu_j K_\varphi(v_j)\right),$$

times the mass counterpart $L^2(M, dM)$.

- Physical states are given by $|\tilde{g}, \vec{k}, M\rangle_{\rm phy}$ with $\tilde{g}$ a diffeo equivalence class of one dimensional graphs, the $\vec{k}$'s proportional to the eigenvalues of the areas of symmetry and M the ADM mass.

Loop quantization of a spherically symmetric spacetime: Observables

- The observables are $\hat{M}$ and $\hat{O}(z)|\tilde{g},\vec{k},M\rangle_{\rm phy} = \ell_{\rm Pl}^2 k_{{\rm Int}(Vz)}|\tilde{g},\vec{k},M\rangle_{\rm phy}$, where ${\rm Int}(Vz)$ is the integer part of Vz.

- Other parametrized Dirac observables can be defined as evolving constants of the motion: their value depends on a gauge parameter whose choice is tantamount to choosing a gauge. This is the case of

$$\hat{E}^x(x)|\tilde{g},\vec{k},M\rangle_{\rm phy} = \hat{O}\big(z(x)\big)|\tilde{g},\vec{k},M\rangle_{\rm phy}, \tag{4}$$

or

$$\hat{g}_{tx} = -\frac{(\hat{E}^x)'\mathcal{K}_\varphi}{2\sqrt{\hat{E}^x}}\frac{1}{\sqrt{1+\mathcal{K}_\varphi^2-\frac{2G\hat{M}}{\sqrt{\hat{E}^x}}}}. \tag{5}$$

Test fields on effective quantum spacetime

- Let us consider an effective geometry with precise values of $\vec{k}$ (growing monotonically and having small differences) and peaked on a given mass M.

- The successive spheres have radius $r_i^2 = \ell_{\text{Planck}}^2 k_i$. The difference between two successive values is at least $\ell_{\text{Planck}}^2/(2r) < \ell_{\text{Planck}}^2/(2M)$ (the lowest possible separation).

- For the sake of simplicity, we will choose a spin net such that $(r_*)_i = \Delta(i + i_H)$, where $\Delta > \ell_{\text{Planck}}^2/(2M)$, $i = 0, 1, \ldots,$ and i_H is the valence of the closest (outer) vertex to the horizon (compatible with the area quantization).

- A scalar field will have support on the vertices of the spin net, i.e., on a lattice (for the radial direction).

Test fields on effective quantum spacetime

- The radial differential equation becomes a difference one. The polarization of the vacuum for the Boulware vacuum is approximated:
a) In the spatial infinity by

$$\langle \phi_B^2(x) \rangle \simeq \int_0^{\pi/\Delta} d\omega \frac{\omega}{4\pi^2} = \frac{1}{8\Delta^2}. \qquad (6)$$

b) Close to the horizon by

$$\langle \phi_B^2(x) \rangle \simeq \int_0^{\pi/\Delta} d\omega \frac{\omega}{4\pi^2(1 - r/r_S)} = \frac{1}{8\Delta^2(1 - r/r_S)} \simeq \frac{1}{8\Delta^2} \frac{r_S}{\delta}, \qquad (7)$$

with $\delta = r_S - \Delta i_H$ (the distance of the last vertex to the horizon).

- In both cases is finite (unless $\Delta = r_S / i_H$).

- How the discrete geometry can affect the predictions of the Casimir Effect?

The Casimir effect: spherical slabs

- Let us consider two concentric spherical slabs of radius r_0 and $r_0 + L$, respectively, and such that $r_0 \gg r_S$ and $r_0 \gg L$.

- The field modes are

$$u_{n,\ell,m}(t, r, \theta, \varphi) = \exp(-i\omega t) R_\ell(\omega|r) Y_{\ell,m}(\theta, \varphi)/(\sqrt{2\pi\omega} r) \tag{8}$$

- The radial functions fulfill the Dirichlet boundary conditions $R(\omega|r_0) = 0 = R(\omega|r_0 + L)$.

- The radial functions are

$$R_{n,\ell}(\omega_{n,\ell}, r) = A_{n,\ell} r^{1/2} \left(J_{\ell+1/2}\left(\omega_{n,\ell} r\right) - \frac{J_{\ell+1/2}\left(\omega_{n,\ell} r_0\right)}{N_{\ell+1/2}\left(\omega_{n,\ell} r_0\right)} N_{\ell+1/2}\left(\omega_{n,\ell} r\right) \right). \tag{9}$$

The Casimir effect: spherical slabs

- The frequencies are discrete $\omega_{n,\ell}$, with $n = 1, 2, \ldots$, but the concrete expression is not known in close form.

- Using $\mathrm{Energy} \propto \frac{1}{2}\sum_n \sum_\ell \omega_{n,\ell}$, (Özcam, 2012), and separating sums for modes with $\ell \to \infty$ and $n \to \infty$,

i) a convergence factor and the (integral representation of the) Cauchy's theorem, the $\ell \to \infty$ contribution is zero, and

ii) the $n \to \infty$ counterpart, by means of the Abel-Plana formula, yields

$$\mathrm{Energy}_{\mathrm{QFT}} = -\frac{\pi^2}{1440L^3} \simeq -\frac{0.0069}{L^3}. \tag{10}$$

The Casimir effect in quantum spacetime

- To compute the force due to the Casimir effect we will need to compute the integral of the expectation value of $T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{6}(\phi')^2 - \frac{1}{3}\phi\ddot{\phi}$.

- The effective geometry of the quantum space time involves $r_j = r_0 + j\Delta$, with $\Delta > \ell^2_{\text{Planck}}/2r_0$ and $N_I\Delta = L$ (with N_I the number of vertices).

- The radial functions satisfy the difference equation

$$\frac{R_{j+1} - 2R_j + R_{j-1}}{\Delta^2} + \omega^2_{n,\ell}R_j - \frac{\ell(\ell+1)}{r_j^2}R_j = 0, \qquad (11)$$

with the boundary conditions $R_{j=0} = 0 = R_{j=N_I}$.

The Casimir effect in quantum spacetime

- In a very good approximation (notice the finite sums in n and ℓ)

$$\phi = \sqrt{\frac{2\pi}{N_I\Delta}} \sum_{n=1}^{N_I-1} \sum_{\ell=0}^{\frac{2\omega_{n,0}r_0}{e}} \sum_{m=-\ell}^{\ell} \left[\frac{a_{n,\ell,m} e^{-i\omega_{n,\ell}t}}{\sqrt{2\pi\omega_{n,\ell}}} \frac{\sin\left(\frac{\pi n j\Delta}{N_I\Delta}\right)}{r_j} Y_{\ell,m}(\theta,\varphi) + \text{c.c.} \right], \tag{12}$$

with

$$\omega_{n,\ell} \simeq \frac{2}{\Delta}\sin\left(\frac{\Delta k_n}{2}\right) + \frac{\ell(\ell+1)}{\frac{4r_0^2}{\Delta}\sin\left(\frac{\Delta k_n}{2}\right)}, \quad k_n = \frac{n\pi}{N_I\Delta}. \tag{13}$$

- The cutoff in n is due to the discretization of the radial coordinate.

- The cutoff in ℓ is motivated by the asymptotic behavior at $\ell \to \infty$ of the Bessel functions.

The Casimir effect in quantum spacetime

- Operating the Green's function $G_+^L(x,x') = \langle 0_L|\phi(x)\phi(x')|0_L\rangle$ we get

$$G_+^L\left(r,t;r',t'\right) \simeq \frac{1}{\pi}\int_{\pi/L}^{\pi/\Delta} dk \int_0^{\frac{2\omega_{k,0}r_0}{e}} d\ell \frac{2\ell}{4\pi\omega_{k,\ell}} \frac{e^{-i\omega_{k,\ell}(t-t')}}{rr'} \sin\left(kz\right)\sin\left(kz'\right) \quad (14)$$

with $z = r - r_0$.

- We have employed $\sum_{m=-\ell}^{\ell} Y_{\ell,m}\left(\theta,\varphi\right) Y_{\ell,m}^*\left(\theta,\varphi\right) = \frac{2\ell+1}{4\pi}$.

- We have replaced sums by integrals (keeping the dominant contributions).

The Casimir effect in quantum spacetime

- In order to compute the stress energy tensor we need

$$F_1^L(z) := \langle 0_L|\dot{\phi}^2|0_L\rangle = \frac{\partial^2}{\partial t\,\partial t'}G_+^L\Big|_{(r,t)=(r',t')} = -\frac{8(1+e^2)r_0^2}{\pi^2 e^4\Delta^3 r^2}\times$$

$$\int_{\pi/L}^{\pi/\Delta} dk\,\sin^3\left(\frac{\Delta k}{2}\right)\sin^2(kz)\,, \tag{15}$$

$$F_2^L(z) := \langle 0_L|\left(\phi'\right)^2|0_L\rangle = \frac{\partial^2}{\partial r\,\partial r'}G_+^L\Big|_{(r,t)=(r',t')} = \frac{[\log(2+e^2)-2]r_0^2}{\pi^2 r^2\Delta}\times$$

$$\int_{\pi/L}^{\pi/\Delta} dk\,\sin\left(\frac{\Delta k}{2}\right)\left(k^2\cos^2(kz) - \frac{2k\cos(kz)\sin(kz)}{r} + \frac{\sin^2(kz)}{r^2}\right), \tag{16}$$

$$\langle 0_L|\phi\ddot{\phi}|0_L\rangle = \frac{\partial^2}{\partial (t')^2}G_+^L\Big|_{(r,t)=(r',t')} = -\langle 0_L|\dot{\phi}^2|0_L\rangle. \tag{17}$$

The Casimir effect in quantum spacetime

- For small lattice step Δ, if $z \neq 0$

$$\langle 0_L | T_{00} | 0_L \rangle = \frac{5}{6} F_1^L(z) + \frac{1}{6} F_2^L(z) \simeq -\frac{a_1}{\Delta^4} \left[1 + O(z/r_0) \right] + \frac{a_2}{\Delta^2 z^2} \left[1 + O(z^2/r_0^2) \right] + \frac{a_3}{L^4} + \frac{a_4}{z^4} + \left(\frac{a_5}{z^4} - \frac{a_6}{L^2 z^2} \right) \cos\left(\frac{2\pi z}{L} \right) + \left(\frac{a_7}{L z^3} - \frac{a_8}{L^3 z} \right) \sin\left(\frac{2\pi z}{L} \right) + O(1/r_0) + O(\Delta), \quad (18)$$

with a_i some positive constants lower than the unit, and with the $O(1/r_0)$ and higher terms finite in the limit $\Delta \to 0$; and for $z = 0$,

$$\langle 0_L | T_{00} | 0_L \rangle \simeq \frac{4(\pi - 2)\left(\log\left[\frac{2}{e^2} + 1 \right] \right)}{3\pi^2 \Delta^4} - \frac{\pi^2 \left(\log\left[\frac{2}{e^2} + 1 \right] \right)}{48 L^4} + O(\Delta^2). \quad (19)$$

Hence, the stress energy tensor remains finite whenever the limit $\Delta \to 0$ is not taken.

The Casimir effect in quantum spacetime

- The Casimir energy is obtained by subtracting the contribution of two slabs separated a distance $L_M >> L$, i.e.,

$$\text{Energy} = -\int_0^L dz \left[\frac{5}{6}\left(F_1^L(z) - F_1^{L_M}(z)\right) + \frac{1}{6}\left(F_2^L(z) - F_2^{L_M}(z)\right)\right] = -\frac{0.15}{L^3} + O(L/L_M) + O(\Delta), \quad (20)$$

- The prediction of QFT for flat slabs is

$$\text{Energy}_{\text{QFT}} = -\frac{\pi^2}{1440L^3} \simeq -\frac{0.0069}{L^3}.$$

- Final remark: the $\ell = 0$ mode is analog to the 1+1 model, and gives the exact result:

$$\text{Energy}_{\text{dim}=1} = -\frac{\pi}{24L} + O(\Delta). \quad (21)$$

Summary

- Techniques of quantum field theory on quantum spherically symmetric vacuum space-time permit to compute the polarization of the vacuum.

- All calculations are naturally regularized by the quantum space-time, yielding finite results.

- The Casimir effect between two concentric spherical plates has been computed, yielding the correct behavior in the separation of the plates with a numerical value that is different of the standard one.

- Better approximations are needed.

- Extension to more general situations.