Matter Couplings

in Coherent States Path Integral Formulation of LQG

April 19, 2022

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Based on arXiv:2101.07659 and arXiv:22xx.xxxxx

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Outline

Standard model (scalar + YM + fermions) matter couplings in full LQG + effective dynamics

AQG framework + coherent states path integral

- Introduction:

  Review reduced phase space framework and classical matter couplings

- Quantum theory and effective dynamics:
  - Discretization and quantization of matter sector
  - Coherent states and Hamiltonian Operator
  - Coherent states path integral and effective dynamics
- Explore semi-classical effective dynamics: some examples
  - Scalar field, inflationary cosmology
  - Fermions
- Symbolic+Numerical Library for effective dynamics (SymPy+SymEngine+Julia)
- Conclusion and Outlook

Introduction

Matter Couplings

Matter couplings at classical level

↓

Solve constraints classically

↓

Reduced phase space framework:
parametrizing gravity variables
with values of dust fields

+

Matter couplings at quantum level?
inflationary cosmology, charged BH, detection of QG effect etc.

Classical matter coupling

- Brown-Kuchar dust

Brown and Kuchar 1994, Giesel and Thiemann 2007

$$S_{BKD}[\rho, g_{\mu\nu}, T, S^j, W_j] \quad = \quad -\frac{1}{2}\int \mathrm{d}^4x\, \sqrt{|\det(g)|}\, \rho\, [g^{\mu\nu}U_\mu U_\nu + 1], \quad U_\mu = -\partial_\mu T + W_j \partial_\mu S^j$$

- Gaussian dust

Kuchar and Torre 1990, Giesel and Thiemann 2015

$$S_{GD}[\rho, g_{\mu\nu}, T, S^j, W_j] \quad = \quad -\int \mathrm{d}^4x\, \sqrt{|\det(g)|}\left[\frac{\rho}{2}\,(g^{\mu\nu}\partial_\mu T \partial_\nu T + 1) + g^{\mu\nu}\partial_\mu T\left(W_j \partial_\nu S^j\right)\right]$$

- Massless real scalar field

Rovelli and Smolin 1993, Domagala, Giesel, Kaminski, and Lewandowski 2010

$$S_\phi\,[g_{\mu\nu}, \phi] = -\frac{1}{2}\int \mathrm{d}^4x \sqrt{|\det(g)|}\, g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi$$

Dirac observables = parametrizing gravity variables with values of dust fields

$T(x) = \tau$	$S^j(x) = \sigma^j$	$O(\tau, \sigma) = O(x)\|_{T(x)=\tau, S^j(x)=\sigma^j}$
physical time variable	physical space variable	Physical observables

Gravity Dirac observables:

Rovelli 2001,Dittrich 2004 ,Thiemann 2004

- SU(2) Ashtekar-Barbero connection: $A(\tau, \sigma) = A(x)|_{T(x)=\tau, S^j(x)=\sigma^j}$,
- triad: $\{E(\tau, \sigma) = E(x)|_{T(x)=\tau, S^j(x)=\sigma^j}$
- canonical structure: $\{E^i_a(\tau, \sigma),\ A^b_j(\tau, \sigma')\} = \frac{1}{2}\kappa\beta\delta^i_j\delta^b_a\delta^3(\sigma, \sigma')$

Solveing total (Abelianized) constraints

$$\mathcal{C}^{tot} = P + h(A, E, \partial_i T) \sim 0, \quad \mathcal{C}_i^{tot} = P_i + (\partial_j S^i)^{-1} (\mathcal{C}_j(A, E) + P\partial_j T) \sim 0$$

Physical Hamiltonian $\mathbf{H}_0 = \int d^3\sigma h$

Giesel and Thiemann 2007, 2015

- Brown-Kuchar dust

$$h = \sqrt{\mathcal{C}(\sigma,\tau)^2 - \frac{1}{4}\sum_{a=1}^{3} \mathcal{C}_a(\sigma,\tau)^2}, \qquad \text{requires } \mathcal{C}(\sigma,\tau)^2 - \frac{1}{4}\sum_{a=1}^{3} \mathcal{C}_a(\sigma,\tau)^2 > 0$$

- Gaussian dust

$h = \mathcal{C}(\sigma,\tau),\qquad \mathcal{C}(\sigma,\tau) < 0$ for physical dust, $\mathcal{C}(\sigma,\tau) > 0$ for phantom dust

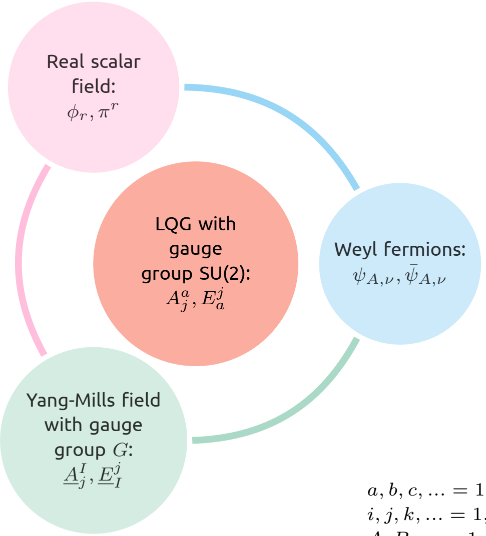
EoMs: Hamilton equation:

$$\frac{dO}{d\tau} = \{\mathbf{H}_0, O\}$$

Note that we use $\{E_a^i(\tau,\sigma),\ A_j^b(\tau,\sigma')\} = \frac{1}{2}\kappa\beta\delta_j^i\delta_a^b\delta^3(\sigma,\sigma')$

a, b, c,... SU(2) indices
i, j, k,... spatial indices of dust space
(σ's with constant τ)

Matter Couplings



Ashtekar, Bianchi, Giesel, Han, Kisielowski, Lewandowski, Ma, Magliaro, Oriti, Perini, Rovelli, Sahlmann, Thiemann, Wieland, Zhang...

Gauge group $SU(2) \times G \in SU(N)$

	SU(2)	G
A_j^a, E_a^j	(adj,adj)	trivial
$\underline{A}_j^I, \underline{E}_I^j$	trivial	(adj,adj)
$\xi_{A,\nu}, \bar{\xi}_{A,\nu}$	$(\mathbf{2}, \mathbf{2})$	$(R_f, \bar{R}_f)$
ϕ_r, π^r	trivial	(R_s, R_s)

Dirac observables:

$$O(\tau, \sigma) = O(x)|_{T(x)=\tau, S^j(x)=\sigma^j}$$

Poisson bracket:

$$\left\{E_a^i(\sigma, \tau), A_j^b\left(\sigma', \tau\right)\right\} = \frac{1}{2}\kappa\beta\delta_j^i\delta_a^b\delta^3\left(\sigma, \sigma'\right)$$

$$\left\{\underline{E}_I^i(\sigma, \tau), \underline{A}_j^J\left(\sigma', \tau\right)\right\} = Q^2\delta_j^i\delta_I^J\delta^3\left(\sigma, \sigma'\right)$$

$$\left\{\xi_{A,\nu}(\sigma, \tau), \bar{\xi}_{B,\rho}\left(\sigma', \tau\right)\right\}_+ = -i\delta_{AB}\delta_{\nu\rho}\delta^3\left(\sigma, \sigma'\right)$$

$$\left\{\pi^r(\sigma, \tau), \phi_s(\sigma, \tau)\right\} = \delta_s^r\delta^3\left(\sigma, \sigma'\right)$$

$a, b, c, \ldots = 1, 2, 3$ SU(2) indices
$i, j, k, \ldots = 1, 2, 3$ spatial indices of dust space (σ's with constant τ)
$A, B, \ldots = 1, 2$ spinor indices of SU(2)
$I, J, K, \ldots = 1, \ldots,$ dim$(R_f(G))$ unitary reps R_f of G
$r = 1, \ldots,$ dim$(R_s(G))$ real reps R_s of G

Classical Hamiltonians

$\mathcal{C}$ and $\mathcal{C}_j$ for gravity + matter in physical Hamiltonian:

$$\begin{aligned}\mathcal{C} &= \mathcal{C}^{GR} + \mathcal{C}^{YM} + \mathcal{C}^{F} + \mathcal{C}^{S},\\ \mathcal{C}_j &= \mathcal{C}_j^{GR} + \mathcal{C}_j^{YM} + \mathcal{C}_j^{F} + \mathcal{C}_j^{S}, \qquad \mathcal{C}_a = 2\mathcal{C}_j e_a^j\end{aligned}$$

The gravity and matter Hamiltonians are given as

	$\mathcal{C}$	$\mathcal{C}_j$
Gravity:	$\mathcal{C}^{GR} = \frac{1}{\kappa}\left[F_{jk}^a - \left(\beta^2+1\right)\varepsilon_{ade}K_j^d K_k^e\right]\varepsilon^{abc}\frac{E_b^j E_c^k}{\sqrt{\det(q)}} + \frac{2\Lambda}{\kappa}\sqrt{\det(q)}$	$\mathcal{C}_j^{GR} = \frac{2}{\kappa\beta}F_{jk}^b E_b^k$
Scalar:	$\mathcal{C}^S = \frac{1}{2\sqrt{\det(q)}}\pi\pi^T + \frac{1}{2}\sqrt{\det(q)}q^{jk}(\mathcal{D}_j\phi)^T\mathcal{D}_k\phi + \sqrt{\det(q)}U(\phi)$	$\mathcal{C}_j^S = \pi\mathcal{D}_j\phi$
YM:	$\mathcal{C}^{YM} = \frac{1}{Q^2}\left[\frac{1}{2}\frac{1}{\sqrt{\det(q)}}q_{ij}\underline{E}_I^i\underline{E}_I^j + \frac{1}{4}\sqrt{\det(q)}q^{ij}q^{kl}\underline{F}_{ik}^I\underline{F}_{jl}^I\right]$	$\mathcal{C}_j^{YM} = \frac{1}{Q^2}\underline{F}_{jk}^I\underline{E}_I^k$
Fermions:	$\mathcal{C}^F = E_a^j\frac{1}{\sqrt{\det(q)}}\Big[-\xi^\dagger\frac{\tau^a}{2}\mathcal{D}_j\xi + (\mathcal{D}_j\xi)^\dagger\frac{\tau^a}{2}\xi + i\beta\mathcal{D}_j\left(\xi^\dagger\frac{\tau^a}{2}\xi\right)$ $- \beta K_j^b\left(\delta_{ab}\xi^\dagger\xi + \frac{i(\beta^2+1)}{\beta}\epsilon_{abc}\left(\xi^\dagger\frac{\tau^c}{2}\xi\right)\right)\Big]$	$\mathcal{C}_j^F = -\frac{i}{2}\left[\xi^\dagger\mathcal{D}_j\xi - (\mathcal{D}_j\xi)^\dagger\xi\right]$

$\mathcal{D}_j$ is the covariant derivative of $\mathrm{SU}(2)\times G$:

$$\mathcal{D}_j\xi = \left[\partial_j + A_j^a\frac{\tau^a}{2} + \underline{A}_j^I T_{R_f}^I\right]\xi, \quad D_j\phi = \left(\partial_j + \underline{A}_j^I T_{R_s}^I\right)\phi.$$

Gauss constraint:

$$\mathcal{G}_a = \frac{1}{\beta\kappa}\mathcal{D}_j E_a^j - \frac{i}{2}\xi^\dagger\frac{\tau_j}{2}\xi = 0$$

$$\underline{\mathcal{G}}_I = \frac{1}{Q^2}\mathcal{D}_a\underline{E}_I^a - \pi T_{R_s}^I\phi - i\xi^\dagger T_{R_f}^I\xi = 0$$

$T_{R_f}^I$,$T_{R_s}^I$ the representation of $\mathfrak{g}$ generators

Quantum theory and effective dynamics

Quantization

The quantization is on a given **fixed** lattice which partitions the dust space

e.g. cubic lattice Γ partitioning 3-torus without boundary

	discretization	quantization
Gravity: A_j^a, E_a^j	$h(e) := \mathcal{P}\exp\int_e A,$ $p^a(e) := -\frac{1}{2\beta a^2}\operatorname{tr}\Big[\tau^a \int_{S_e} \varepsilon_{ijk}\mathrm{d}\sigma^i \wedge \mathrm{d}\sigma^j$ $h\left(\rho_e(\sigma)\right) E_b^k(\sigma)\tau^b\, h\left(\rho_e(\sigma)\right)^{-1}\Big]$ $(h(e), p^a(e))$ holonomy-flux algebra	$\mathcal{H}_\gamma^{GR} \simeq L^2(SU(2), \mathrm{d}\mu_H)^{\otimes\lvert E(\gamma)\rvert}$, $\hat{h}(e)$ multiplication operator, $\hat{p}^a(e) = i l_p^2 R_e^a$ where $\hat{R}_e^a$ right invariant vector field on SU(2)

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YM: $\underline{A}_j^I, \underline{E}_I^j$	$\underline{h}(e) := \mathcal{P}\exp\int_e \mathrm{d}\sigma^j \underline{A}_j^I T^I,$ $\underline{p}^I(e) := -2\mathrm{tr}\Big[T^I \int_{S_e} \varepsilon_{ijk}\mathrm{d}\sigma^i \wedge \mathrm{d}\sigma^j$ $\underline{h}(\rho_e(\sigma))\, \underline{E}_J^k(\sigma) T^J\, \underline{h}(\rho_e(\sigma))^{-1}\Big]$ $(\underline{h}(e), \underline{p}^I(e))$ holonomy-flux algebra	$\mathcal{H}_\gamma^{YM} \simeq L^2(G, \mathrm{d}\mu_H)^{\otimes\|E(\gamma)\|}$, $\hat{\underline{h}}(e)$ multiplication operator, $\hat{\underline{p}}^I(e) = i\hbar Q^2 \hat{\underline{R}}_e^I$ where $\hat{\underline{R}}_e^I$ right invariant vector field on G

Quantization

The quantization is on a given **fixed** lattice which partitions the dust space

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	discretization	quantization
Gravity: A^a_j, E^j_a	$h(e) := \mathcal{P}\exp\int_e A,$ $p^a(e) := -\frac{1}{2\beta a^2}\mathrm{tr}\Big[\tau^a \int_{S_e} \varepsilon_{ijk}\mathrm{d}\sigma^i \wedge \mathrm{d}\sigma^j$ $h\left(\rho_e(\sigma)\right)\ E^k_b(\sigma)\tau^b\ h\left(\rho_e(\sigma)\right)^{-1}\Big]$ $(h(e), p^a(e))$ holonomy-flux algebra	$\mathcal{H}^{GR}_\gamma \simeq L^2(SU(2), \mathrm{d}\mu_H)^{\otimes\lvert E(\gamma)\rvert}$, $\hat{h}(e)$ multiplication operator, $\hat{p}^a(e) = i l_p^2 R^a_e$ where $\hat{R}^a_e$ right invariant vector field on SU(2)
YM: $\underline{A}^I_j, \underline{E}^j_I$	$\underline{h}(e) := \mathcal{P}\exp\int_e \mathrm{d}\sigma^j \underline{A}^I_j T^I,$ $\underline{p}^I(e) := -2\mathrm{tr}\Big[T^I \int_{S_e} \varepsilon_{ijk}\mathrm{d}\sigma^i \wedge \mathrm{d}\sigma^j$ $\underline{h}\left(\rho_e(\sigma)\right)\ \underline{E}^k_J(\sigma)T^J\ \underline{h}\left(\rho_e(\sigma)\right)^{-1}\Big]$ $(\underline{h}(e), \underline{p}^I(e))$ holonomy-flux algebra	$\mathcal{H}^{YM}_\gamma \simeq L^2(G, \mathrm{d}\mu_H)^{\otimes\lvert E(\gamma)\rvert}$, $\hat{\underline{h}}(e)$ multiplication operator, $\hat{\underline{p}}^I(e) = i\hbar Q^2 \hat{\underline{R}}^I_e$ where $\hat{\underline{R}}^I_e$ right invariant vector field on G
Fermions: $\xi_{A,\nu}, \bar{\xi}_{A,\nu}$	$\theta_{A,\mu}(v) := \int_\Sigma \mathrm{d}^3x \frac{\chi_\mu(x,v)}{\sqrt{\mu^3}}\xi_{A,\mu}(y)$ $\left\{\theta_{A,\nu}(v), \bar{\theta}_{B,\rho}(v')\right\}_+ = -i\delta_{\nu\rho}\delta_{AB}\delta_{v,v'}$	$\mathcal{H}^F_\gamma = \otimes_{v\in V(\gamma)}\mathcal{H}^F_v, \quad \mathcal{H}^F_v = (\mathbb{C}^2)^{2\dim(R_f)}$ $\left[\hat{\theta}_{A,\nu}(v), \hat{\bar{\theta}}_{B,\rho}(v)\right]_+ = \hbar\delta_{\mu\rho}\delta^{AB}\delta_{v,v'}$ $\hat{\theta}_{A,\nu}(v)f(\theta) = \theta_{A,\nu}(v)f(\theta),\ \hat{\bar{\theta}}_{A,\nu}(v)f(\theta) = \hbar\left[\partial/\partial\theta_{A,\nu}(v)\right]f(\theta)$

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Gravity: A_j^a, E_a^j	$h(e) := \mathcal{P}\exp\int_e A,$ $p^a(e) := -\frac{1}{2\beta a^2}\mathrm{tr}\Big[\tau^a \int_{S_e} \varepsilon_{ijk}\mathrm{d}\sigma^i \wedge \mathrm{d}\sigma^j$ $h(\rho_e(\sigma))\; E_b^k(\sigma)\tau^b\; h(\rho_e(\sigma))^{-1}\Big]$ $(h(e), p^a(e))$ holonomy-flux algebra	$\mathcal{H}_\gamma^{GR} \simeq L^2(SU(2), \mathrm{d}\mu_H)^{\otimes \lvert E(\gamma)\rvert}$, $\hat{h}(e)$ multiplication operator, $\hat{p}^a(e) = il_p^2 R_e^a$ where $\hat{R}_e^a$ right invariant vector field on SU(2)
YM: $\underline{A}_j^I, \underline{E}_I^j$	$\underline{h}(e) := \mathcal{P}\exp\int_e \mathrm{d}\sigma^j \underline{A}_j^I T^I,$ $\underline{p}^I(e) := -2\mathrm{tr}\Big[T^I \int_{S_e} \varepsilon_{ijk}\mathrm{d}\sigma^i \wedge \mathrm{d}\sigma^j$ $\underline{h}(\rho_e(\sigma))\; \underline{E}_J^k(\sigma)T^J\; \underline{h}(\rho_e(\sigma))^{-1}\Big]$ $(\underline{h}(e), \underline{p}^I(e))$ holonomy-flux algebra	$\mathcal{H}_\gamma^{YM} \simeq L^2(G, \mathrm{d}\mu_H)^{\otimes \lvert E(\gamma)\rvert}$, $\hat{\underline{h}}(e)$ multiplication operator, $\hat{\underline{p}}^I(e) = i\hbar Q^2 \hat{\underline{R}}_e^I$ where $\hat{\underline{R}}_e^I$ right invariant vector field on G
Fermions: $\xi_{A,\nu}, \bar{\xi}_{A,\nu}$	$\theta_{A,\mu}(v) := \int_\Sigma \mathrm{d}^3x \frac{\chi_\mu(x,v)}{\sqrt{\mu^3}}\xi_{A,\mu}(y)$ $\{\theta_{A,\nu}(v), \bar{\theta}_{B,\rho}(v')\}_+ = -i\delta_{\nu\rho}\delta_{AB}\delta_{v,v'}$	$\mathcal{H}_\gamma^F = \otimes_{v\in V(\gamma)}\mathcal{H}_v^F, \quad \mathcal{H}_v^F = (\mathbb{C}^2)^{2\dim(R_f)}$ $\Big[\hat{\theta}_{A,\nu}(v), \hat{\bar{\theta}}_{B,\rho}(v)\Big]_+ = \hbar\delta_{\mu\rho}\delta^{AB}\delta_{v,v'}$ $\hat{\theta}_{A,\nu}(v)f(\theta) = \theta_{A,\nu}(v)f(\theta),\ \hat{\bar{\theta}}_{A,\nu}(v)f(\theta) = \hbar\left[\partial/\partial\theta_{A,\nu}(v)\right]f(\theta)$
Scalar fields: ϕ_r, π^r	$\phi(v)$ and $\pi(v) = \int \mathrm{d}^3x\chi_\mu(x,v)\pi(x)$ $\{\pi^r(v), \phi_r(v')\} = \delta_{v,v'}.$	$\mathcal{H}_\gamma^S = \otimes_{v\in V(\gamma)}\mathcal{H}_v, \mathcal{H}_v \simeq L^2(\mathbb{R}^{\dim(R_s)}, \prod^r \mathrm{d}\phi_r(v))$ $\Big[\hat{\pi}(v), \hat{\phi}(v')\Big] = i\hbar\delta_{v,v'}$ $\hat{\phi}(v)f(\phi) = \phi(v)f(\phi), \quad \hat{\pi}(v)f(\phi) = i\hbar\left[\partial/\partial\phi(v)\right]f(\phi)$

Quantization

Kinematic Hilbert space of gravity coupled to matters

$$\mathcal{H}^0_\gamma = \mathcal{H}^{GR}_\gamma \otimes \mathcal{H}^{YM}_\gamma \otimes \mathcal{H}^F_\gamma \otimes \mathcal{H}^S_\gamma$$

Gauge transformations of $\mathrm{SU}(2) \times G$ given by Gauss constraint:

$$\hat{U}_u : f\left(h(e), \underline{h}(e), \theta(v), \phi(v)\right) \quad \mapsto \quad f^u\left(h(e), \underline{h}(e), \theta(v), \phi(v)\right) = f\left(h(e)^u, \underline{h}(e)^u, \theta(v)^u, \phi(v)^u\right)$$

where

$$h(e)^u = u_{s(e)} h(e) u^{-1}_{t(e)}, \quad \underline{h}(e)^u = \underline{u}_{s(e)} \underline{h}(e) \underline{u}^{-1}_{t(e)},$$
$$\theta(v)^u = (u_v \otimes R_f(\underline{u}_v))\theta(v), \quad \phi(v)^u = R_s(\underline{u}_v)\phi(v).$$

Physical Hilbert space by group averaging: $f \in \mathcal{H}^0_\gamma \to f_{inv} \in \mathcal{H}_\gamma$

$$f_{inv} = \int_{(\mathrm{SU}(2)\times G)^{|V(\gamma)|}} \prod_{v \in V(\gamma)} \mathrm{d}\mu_H(u_v) \mathrm{d}\mu_H(\underline{u}_v)\, f^u \in \mathcal{H}_\gamma .$$

Coherent States

Gravity:
θ^j_a,p^a_j

$$\psi^t_g = \prod_{e\in E(\gamma)} \psi^t_{g(e)}, \quad \psi^t_{g(e)}(h(e)) = \sum_{j_e\in\mathbb{Z}_+/2\cup\{0\}} (2j_e+1)\, e^{-tj_e(j_e+1)/2} \chi_{j_e}\left(g(e)h(e)^{-1}\right)$$

$g(e) = e^{-ip^a(e)\tau^a/2} e^{\theta^a(e)\tau^a/2}, \quad p^a(e),\ \theta^a(e) \in \mathbb{R}^3$ parametrize gravity sector

Coherent States

Gravity: θ_a^j,p_j^a	$\psi_g^t = \prod_{e\in E(\gamma)} \psi^t_{g(e)}, \quad \psi^t_{g(e)}(h(e)) = \sum_{j_e\in\mathbb{Z}_+/2\cup\{0\}} (2j_e+1)\, e^{-tj_e(j_e+1)/2} \chi_{j_e}\left(g(e)h(e)^{-1}\right)$ $g(e) = e^{-ip^a(e)\tau^a/2} e^{\theta^a(e)\tau^a/2}, \quad p^a(e),\ \theta^a(e) \in \mathbb{R}^3$ parametrize gravity sector
YM with $G \in SU(N)$: $\underline{\theta}_j^I, \underline{p}_I^j$	$\psi^t_{\underline{g}} = \prod_{e\in E(\gamma)} \psi^t_{\underline{g}(e)}, \quad \psi^t_{\underline{g}(e)}(\underline{h}(e)) = \sum_{R\in\mathrm{Irrep}(G)} \dim(R)\, e^{-2\hbar Q^2\lambda_R/2} \chi_R\left(\underline{g}(e)\underline{h}(e)^{-1}\right)$ $\underline{g}(e) = e^{-i\underline{p}^I(e)T^I} e^{\underline{\theta}^I(e)T^I}, \quad \underline{p}^I(e),\ \underline{\theta}^I(e) \in \mathbb{R}^{\dim(G)}$ parametrize YM sector

Coherent States

Gravity: θ_a^j,p_j^a	$\psi_g^t = \prod_{e\in E(\gamma)} \psi_{g(e)}^t, \quad \psi_{g(e)}^t(h(e)) = \sum_{j_e\in\mathbb{Z}_+/2\cup\{0\}} (2j_e+1)\, e^{-tj_e(j_e+1)/2} \chi_{j_e}\left(g(e)h(e)^{-1}\right)$ $g(e) = e^{-ip^a(e)\tau^a/2} e^{\theta^a(e)\tau^a/2}, \quad p^a(e),\ \theta^a(e) \in \mathbb{R}^3$ parametrize gravity sector
YM with $G \in SU(N)$: $\underline{\theta}_j^I, \underline{p}_I^j$	$\psi_{\underline{g}}^t = \prod_{e\in E(\gamma)} \psi_{\underline{g}(e)}^t, \quad \psi_{\underline{g}(e)}^t(\underline{h}(e)) = \sum_{R\in\mathrm{Irrep}(G)} \dim(R)\, e^{-2\hbar Q^2\lambda_R/2} \chi_R\left(\underline{g}(e)\underline{h}(e)^{-1}\right)$ $\underline{g}(e) = e^{-i\underline{p}^I(e)T^I} e^{\underline{\theta}^I(e)T^I}, \quad \underline{p}^I(e),\ \underline{\theta}^I(e) \in \mathbb{R}^{\dim(G)}$ parametrize YM sector
Fermions: $\xi_{A,\nu}, \bar{\xi}_{A,\nu}$	$\lvert\psi_\zeta^\hbar\rangle = \otimes_{v,A,\nu} e^{-\frac{1}{\hbar}\bar{\zeta}_{A,\nu}(v)\hat{\theta}_{A,\nu}(v)}\lvert 0\rangle,$ or $\psi_\zeta^\hbar(\theta) = \prod_v e^{-\frac{1}{\hbar}\sum_{A,\nu}\bar{\zeta}_{A,\nu}(v)\theta_{A,\nu}(v)}$ $\zeta(v), \bar{\zeta}(v)$ parametrize fermions sector

Coherent States

Gravity: θ_a^j,p_j^a	$\psi_g^t = \prod_{e\in E(\gamma)} \psi_{g(e)}^t, \quad \psi_{g(e)}^t(h(e)) = \sum_{j_e\in\mathbb{Z}_+/2\cup\{0\}} (2j_e+1)\, e^{-tj_e(j_e+1)/2}\chi_{j_e}\left(g(e)h(e)^{-1}\right)$ $g(e) = e^{-ip^a(e)\tau^a/2}e^{\theta^a(e)\tau^a/2}, \quad p^a(e),\ \theta^a(e) \in \mathbb{R}^3$ parametrize gravity sector
YM with $G \in SU(N)$: $\underline{\theta}_j^I, \underline{p}_I^j$	$\psi_{\underline{g}}^t = \prod_{e\in E(\gamma)} \psi_{\underline{g}(e)}^t, \quad \psi_{\underline{g}(e)}^t(\underline{h}(e)) = \sum_{R\in\mathrm{Irrep}(G)} \dim(R)\, e^{-2\hbar Q^2\lambda_R/2}\chi_R\left(\underline{g}(e)\underline{h}(e)^{-1}\right)$ $\underline{g}(e) = e^{-i\underline{p}^I(e)T^I}e^{\underline{\theta}^I(e)T^I}, \quad \underline{p}^I(e),\ \underline{\theta}^I(e) \in \mathbb{R}^{\dim(G)}$ parametrize YM sector
Fermions: $\xi_{A,\nu}, \bar{\xi}_{A,\nu}$	$\lvert\psi_\zeta^\hbar\rangle = \otimes_{v,A,\nu} e^{-\frac{1}{\hbar}\bar{\zeta}_{A,\nu}(v)\hat{\theta}_{A,\nu}(v)}\lvert 0\rangle$, or $\psi_\zeta^\hbar(\theta) = \prod_v e^{-\frac{1}{\hbar}\sum_{A,\nu}\bar{\zeta}_{A,\nu}(v)\theta_{A,\nu}(v)}$ $\zeta(v), \bar{\zeta}(v)$ parametrize fermions sector
Scalar fields: ϕ_r, π^r	$\hat{\mathfrak{a}}_r(v)\lvert\psi_z^\hbar\rangle = \frac{z_r(v)}{\sqrt{\hbar}}\lvert\psi_z^\hbar\rangle, \quad \lvert\psi_z^\hbar\rangle = \prod_{v,r} e^{\frac{1}{\sqrt{\hbar}} z_r(v)\hat{\mathfrak{a}}_r(v)^\dagger}\lvert 0\rangle$ $z_r(v) = \frac{1}{\sqrt{2}}\left[\phi_r(v) - i\pi^r(v)\right]$ parametrize scalar sector

Properties of Coherent States

- Coherent states in $\mathcal{H}^0_\gamma$:tensor product over all sectors:

$$\psi^\hbar_Z = \psi^\hbar_g \otimes \psi^\hbar_{\underline{g}} \otimes \psi^\hbar_\zeta \otimes \psi^\hbar_z, \quad Z \equiv (g, \underline{g}, \zeta, z) : \text{parametrization of gravity+matter phase space}$$

- Normalized coherent states: $\tilde{\psi}^\hbar_Z := \psi^\hbar_Z / ||\psi^\hbar_Z||$
- Overlapping function:

$$\langle \psi^\hbar_{Z'} | \psi^\hbar_Z \rangle \quad := \quad \nu(g)\nu(\underline{g})\, e^{\frac{1}{t} K(g_2(e), g_1(e))} e^{\frac{1}{2\hbar Q^2} \underline{K}\left(\underline{g}_2(e), \underline{g}_1(e)\right)} \times$$
$$e^{\frac{1}{\hbar} \sum_v \left[\bar{\zeta}'^\dagger(v)\bar{\zeta}(v) - \frac{1}{2}\bar{\zeta}'^\dagger(v)\bar{\zeta}'(v) - \frac{1}{2}\bar{\zeta}^\dagger(v)\bar{\zeta}(v)\right]} e^{\frac{1}{\hbar} \sum_v \left[z'(v)^\dagger z(v) - \frac{1}{2} z(v)^\dagger z(v) - \frac{1}{2} z'(v)^\dagger z'(v)\right]}$$

- $\tilde{\psi}^\hbar_Z$ satisfies the over-completeness relation

$$\int \mathrm{d}Z |\psi^\hbar_Z\rangle\langle\psi^\hbar_Z| = 1_{\mathcal{H}^0_\gamma}, \quad \mathrm{d}Z = \prod_e \mathrm{d}g(e) \prod_e \mathrm{d}\underline{g}(e) \prod_{v,A,\nu} \left[\hbar \mathrm{d}\bar{\zeta}_{A,\nu}(v) \mathrm{d}\zeta_{A,\nu}(v)\right] \prod_{v,r} \frac{\mathrm{d}^2 z_r(v)}{\pi\hbar}.$$

- gauge transformation

$$\psi^\hbar_g \to \psi^\hbar_{g^u},\ \psi^t_{\underline{g}} \to \psi^t_{\underline{g}^u}, \qquad \text{where} \quad g^u(e) = u^{-1}_{s(e)} g(e) u_{t(e)},\ \underline{g}^u(e) = \underline{u}^{-1}_{s(e)} \underline{g}(e) \underline{u}_{t(e)}$$

$$\psi^\hbar_\zeta \to \psi^\hbar_{\zeta^u}, \qquad \text{where} \quad \zeta^u(v) = \left(u_v^{-1} \otimes R_f\left(\underline{u}_v^{-1}\right)\right)\zeta(v).$$

$$\psi^\hbar_z \to \psi^\hbar_{z^u}, \qquad \text{where} \quad z^u(v) = R_s(\underline{u}_v^{-1}) z(v).$$

- Gauge invariant coherent states $\Psi^\hbar_{[Z]} \in \mathcal{H}_\gamma$ are defined by group averaging

$$\Psi^\hbar_{[Z]} = \int_{(\mathrm{SU}(2)\times G)^{|V(\gamma)|}} \prod_{v\in V(\gamma)} \mathrm{d}\mu_H(u_v) \mathrm{d}\mu_H(\underline{u}_v)\, \psi^\hbar_{Z^u}, \quad \text{where} \quad Z^u \equiv (g^u, \underline{g}^u, \zeta^u, z^u).$$

Hamiltonian operator

Gravity sector:

Quantizing $\mathrm{sgn}(e)\mathcal{C}^{GR}$ and $\mathrm{sgn}(e)\mathcal{C}^{GR}_a$ with Thiemann's trick:

$$\hat{C}_{\mu,v} = -\frac{4}{i\beta^2\kappa\ell_p^2}\sum_{s_1,s_2,s_3=\pm1} s_1 s_2 s_3\, \varepsilon^{I_1 I_2 I_3}\, \mathrm{Tr}\left(\tau^\mu \hat{h}(\alpha_{v;I_1 s_1,I_2 s_2})\hat{h}(e_{v;I_3 s_3})\left[\hat{h}(e_{v;I_3 s_3})^{-1}, \hat{V}_v\right]\right)$$

$$\hat{C}_v = \hat{C}_{0,v} + \frac{1+\beta^2}{2}\hat{C}_{L,v} + \frac{2\Lambda}{\kappa}\hat{V}_v, \qquad \hat{K} = \frac{i}{\hbar\beta^2}\left[\sum_{v\in V(\gamma)}\hat{C}_{0,v}, \sum_{v\in V(\gamma)} V_v\right]$$

$$\hat{C}_{L,v} = \frac{8}{\kappa\left(i\beta\ell_p^2\right)^3}\sum_{s_1,s_2,s_3=\pm1} s_1 s_2 s_3\, \varepsilon^{I_1 I_2 I_3}$$

$$\mathrm{Tr}\left(\hat{h}(e_{v;I_1 s_1})\left[\hat{h}(e_{v;I_1 s_1})^{-1}, \hat{K}\right]\hat{h}(e_{v;I_2 s_2})\left[\hat{h}(e_{v;I_2 s_2})^{-1}, \hat{K}\right]\hat{h}(e_{v;I_3 s_3})\left[\hat{h}(e_{v;I_3 s_3})^{-1}, \hat{V}_v\right]\right).$$

We need to quantize $\mathrm{sgn}(e)\mathcal{C}$ and $\mathrm{sgn}(e)\mathcal{C}_a = 2\mathrm{sgn}(e)\mathcal{C}_j e^j_a$ for all matter sectors!

Giesel and Thiemann 06, 07

Scalar field as an example:

discretizing $\int_{\square} d^3x \,\mathrm{sgn}(e)\mathcal{C}^S$

$$\mathrm{sgn}(e)\mathcal{C}^S = \boxed{\frac{\mathrm{sgn}(e)}{2\sqrt{\det(q)}}}\,\boxed{\pi\pi^T} + \boxed{\frac{1}{2}\mathrm{sgn}(e)\sqrt{\det(q)}q^{jk}}(\mathcal{D}_j\phi)^T\boxed{\mathcal{D}_k\phi} + \boxed{\mathrm{sgn}(e)\sqrt{\det(q)}}U_1(\phi) + \boxed{\sqrt{\det(q)}}U_2(\phi)$$

with: $\frac{\mathrm{sgn}(e)}{2\sqrt{\det(q)}} \to \frac{1}{2}\left(\frac{\mathrm{sgn}(e)}{V}\right)$; $\pi\pi^T \to \pi(v)\pi(v)^T$; $\frac{1}{2}\mathrm{sgn}(e)\sqrt{\det(q)}q^{jk} \to \sim \frac{1}{2}\left(\frac{\mathrm{sgn}(e)}{V}\right)p_a^j p_b^k \delta^{ab}$; $\mathcal{D}_k\phi \to R_s\left(\underline{h}(e_{v;j})\right)\phi\left(t(e_{v;j})\right) - \phi(v) \equiv \delta_j^{(R_s)}\phi(v)$; $\mathrm{sgn}(e)\sqrt{\det(q)} \to \mathrm{sgn}(e)V$; $\sqrt{\det(q)} \to V(v)$

Vector part:

$$\mathrm{sgn}(e)\mathcal{C}_a = 2\mathrm{sgn}(e)\mathcal{C}_j e_a^j = \frac{\mathcal{C}_j}{\sqrt{\det(q)}}\epsilon^{jmn}\epsilon_{abc}e_m^b e_n^c\!: \quad \hat{C}_{a,v} = \left(\frac{32}{\ell_P^4\beta^2}\right)\sum_{s_1s_2s_3} s_1s_2s_3\epsilon_{abc}\epsilon^{jmn}\hat{\mathcal{Q}}^b_{1/2}(e_{v;m,s_m})\widehat{\mathcal{C}_{js_j,v}}\,\hat{\mathcal{Q}}^c_{1/2}(e_{v;ns_n}).$$

We then define

$$\hat{C}_v^S = \frac{1}{2}\left(\widehat{\frac{\mathrm{sgn}(e)}{V}}\right)_v \hat{\pi}(v)^2 + \frac{1}{2}\left(\widehat{\frac{\mathrm{sgn}(e)}{V}}\right)_v \frac{a^4\beta^2}{8}\sum_{s_1s_2s_3}\sum_{j,k} s_jX_a^j(v)s_kX_a^k(v)\left(\delta_{j,s_j}^{(R_s)}\hat{\phi}(v)\right)\left(\delta_{k,s_k}^{(R_s)}\hat{\phi}(v)\right)$$
$$+ (\widehat{\mathrm{sgn}(e)V})_v U_1(\hat{\phi}) + \hat{V}_v U_2(\hat{\phi})$$

$$\hat{C}^S_{js,v} = \hat{\pi}(v)\delta_{j,s}^{(R_s)}\hat{\phi}(v)$$

with:

$$\widehat{\mathrm{sgn}(e)}_v = -\left(\frac{9\times 16}{\ell_P^6\beta^3}\right)\sum_{s_1s_2s_3} s_1s_2s_3\epsilon^{ijk}\mathrm{Tr}\left(\hat{\mathcal{Q}}_{2/3}(e_{v;is_1})\hat{\mathcal{Q}}_{2/3}(e_{v;js_2})\hat{\mathcal{Q}}_{2/3}(e_{v;ks_3})\right)$$

$$\left(\widehat{\frac{\mathrm{sgn}(e)}{V}}\right)_v = -\left(\frac{18\times 64}{\ell_P^6\beta^3}\right)\sum_{s_1s_2s_3} s_1s_2s_3\sum_{i,j,k}\epsilon^{ijk}\mathrm{Tr}\left[\hat{\mathcal{Q}}_{1/3}(e_{v;is_1})\hat{\mathcal{Q}}_{1/3}(e_{v;js_2})\hat{\mathcal{Q}}_{1/3}(e_{v;ks_3})\right]$$

$$(\widehat{\mathrm{sgn}(e)V})_v = -\frac{2}{3}\frac{8^2}{(\ell_P^2\beta)^3}\sum_{s_1s_2s_3} s_1s_2s_3\epsilon^{ijk}\mathrm{Tr}\left[\hat{\mathcal{Q}}_1(e_{v;is_1})\hat{\mathcal{Q}}_1(e_{v;js_2})\hat{\mathcal{Q}}_1(e_{v;ks_3})\right]$$

$$\hat{\mathcal{Q}}^a_r(e) = i\mathrm{Tr}\left(\tau^a\hat{h}(e)\left[\hat{h}(e)^{-1},\hat{V}_v^r\right]\right),\ \hat{\mathcal{Q}}_r(e) = \hat{\mathcal{Q}}^a_r(e)\frac{\tau^a}{2} = -i\hat{h}(e)\left[\hat{h}(e)^{-1},\hat{V}_v^r\right]$$

essentially self-adjoint operators [Sahlmann and Thiemann, 02']

Apply to YM and Fermions: we get

YM

$$\hat{C}_v^{YM} = \frac{1}{Q^2}\widehat{\mathrm{sgn}(e)}_v \left(\hat{C}_{E,v}^{YM} + \hat{C}_{B,v}^{YM}\right)$$

$$\hat{C}_{js_j,v}^{YM} = -\frac{2}{Q^2}\sum_k \frac{\mathrm{Tr}\left[T^I \underline{\hat{h}}(\alpha_{v;js_j,ks_k})\right] s_k \underline{\hat{X}}_I^k(v) + s_k \underline{\hat{X}}_I^k(v)\mathrm{Tr}\left[T^I \underline{\hat{h}}(\alpha_{v;js_j,ks_k})\right]^\dagger}{2}$$

$$\hat{C}_{E,v}^{YM} = \left(\frac{16}{\ell_P^4\beta^2}\right)\sum_{s_1s_2s_3}\left(\sum_{i=1}^3 s_i \underline{\hat{X}}_I^i(v)\hat{\mathcal{Q}}_{1/2}^a(e_{v;is_i})\right)\left(\sum_{j=1}^3 \hat{\mathcal{Q}}_{1/2}^a(e_{v;js_j}) s_j \underline{\hat{X}}_I^j(v)\right),$$

$$\hat{C}_{B,v}^{YM} = \left(\frac{16}{\ell_P^4\beta^2}\right)\sum_{s_1s_2s_3}\sum_{i,k,l}\varepsilon^{ikl}\sum_{j,m,n}\varepsilon^{jmn}\mathrm{Tr}\left(T^I \underline{\hat{h}}(\alpha_{v;ks_k,ls_l})\right)^\dagger \hat{\mathcal{Q}}_{1/2}^a(e_{v;is_i})\hat{\mathcal{Q}}_{1/2}^a(e_{v;js_j})\mathrm{Tr}\left(T^I \underline{\hat{h}}(\alpha_{v;ms_m,ns_n})\right)$$

Hamiltonian operator

Apply to YM and Fermions: we get

YM

$$\hat{C}^{YM}_v = \frac{1}{Q^2}\widehat{\mathrm{sgn}(e)}_v\left(\hat{C}^{YM}_{E,v}+\hat{C}^{YM}_{B,v}\right)$$

$$\hat{C}^{YM}_{js_j,v} = -\frac{2}{Q^2}\sum_k \frac{\mathrm{Tr}\left[T^I\underline{\hat{h}}(\alpha_{v;js_j,ks_k})\right]s_k\underline{\hat{X}}^k_I(v)+s_k\underline{\hat{X}}^k_I(v)\mathrm{Tr}\left[T^I\underline{\hat{h}}(\alpha_{v;js_j,ks_k})\right]^\dagger}{2}$$

$$\hat{C}^{YM}_{E,v} = \left(\frac{16}{\ell_P^4\beta^2}\right)\sum_{s_1s_2s_3}\left(\sum_{i=1}^{3}s_i\underline{\hat{X}}^i_I(v)\hat{\mathscr{Q}}^a_{1/2}(e_{v;is_i})\right)\left(\sum_{j=1}^{3}\hat{\mathscr{Q}}^a_{1/2}(e_{v;js_j})s_j\underline{\hat{X}}^j_I(v)\right),$$

$$\hat{C}^{YM}_{B,v} = \left(\frac{16}{\ell_P^4\beta^2}\right)\sum_{s_1s_2s_3}\sum_{i,k,l}\varepsilon^{ikl}\sum_{j,m,n}\varepsilon^{jmn}\mathrm{Tr}\left(T^I\underline{\hat{h}}(\alpha_{v;ks_k,ls_l})\right)^\dagger\hat{\mathscr{Q}}^a_{1/2}(e_{v;is_i})\hat{\mathscr{Q}}^a_{1/2}(e_{v;js_j})\mathrm{Tr}\left(T^I\underline{\hat{h}}(\alpha_{v;ms_m,ns_n})\right)$$

Fermions:

$$\hat{C}^F_v = \left(\frac{\widehat{\mathrm{sgn}(e)}}{V}\right)_v\frac{a^2\beta}{8}\sum_{s_1s_2s_3}\sum_j s_j\Bigg[\hat{X}^j_a(v)\Bigg(-\hat{\mathscr{D}}^a(e_{v;js_j})+i\beta\hat{\mathscr{V}}^a(e_{v;js_j})$$
$$+\frac{2}{i\ell_P^2}\mathrm{Tr}\left(\tau^b\hat{h}(e_{v;js_j})\left[\hat{h}(e_{v;js_j}),\hat{K}\right]\right)\left[\delta_{ab}\hat{\theta}^\dagger(v)\hat{\theta}(v)+\frac{i\left(\beta^2+1\right)}{\beta}\epsilon_{abc}\hat{\theta}^\dagger(v)\frac{\tau^c}{2}\hat{\theta}(v)\right]\Bigg)\Bigg]$$

$$\hat{C}^F_{j,v} = -\frac{i}{2}\mathscr{D}(e_{v;js_j})$$

$$\hat{\mathscr{D}}^a(e) = \hat{\theta}^\dagger(v)\frac{\tau^a}{2}\hat{h}(e)R_f\left(\underline{\hat{h}}(e)\right)\hat{\theta}(t(e)) - \hat{\theta}^\dagger(t(e))R_f\left(\underline{\hat{h}}(e)^{-1}\right)\hat{h}(e)^{-1}\frac{\tau^a}{2}\hat{\theta}(v),$$

$$\mathscr{D}(e) = \hat{\theta}^\dagger(v)\hat{h}(e)R_f\left(\underline{\hat{h}}(e)\right)\hat{\theta}(t(e)) - \hat{\theta}^\dagger(t(e))R_f\left(\underline{\hat{h}}(e)^{-1}\right)\hat{h}(e)^{-1}\hat{\theta}(v)$$

$$\hat{\mathscr{V}}^a(e) = \hat{\theta}^\dagger(t(e))\hat{h}(e)^{-1}\frac{\tau^a}{2}\hat{h}(e)\hat{\theta}(t(e)) - \hat{\theta}^\dagger(v)\frac{\tau^a}{2}\hat{\theta}(v).$$

Physical Hamiltonian: Summary

- Scalar and Vector part of Hamiltnian

$$\hat{C}_v = \hat{C}_v^{GR} + \hat{C}_v^{YM} + \hat{C}_v^{F} + \hat{C}_v^{S},$$
$$\hat{C}_{a,v} = \hat{C}_{a,v}^{GR} + \hat{C}_{a,v}^{YM} + \hat{C}_{a,v}^{F} + \hat{C}_{a,v}^{S}.$$

- Physical Hamiltonian
  - Brown-Kuchar dust: $\mathbf{H} = \sum_{v \in V(\gamma)} H_v, \quad H_v = \sqrt{\left| C_v^2 - \frac{\alpha}{4} \sum_{a=1}^{3} C_{a,v}^2 \right|}$
  - Gaussian dust: $\mathbf{H} = \sum_{v \in V(\gamma)} H_v, \quad H_v = C_v$
- self-adjoint non-graph changing Hamiltonian $\hat{\mathbf{H}} = \sum_{v \in V(\gamma)} \hat{H}_v$:
  - Brown-Kuchar dust

$$\hat{H}_v := \left[\hat{M}_-^{\dagger}(v) \hat{M}_-(v) \right]^{1/4}, \quad \hat{M}_-(v) = \hat{C}_v^{\dagger} \hat{C}_v - \frac{1}{4} \sum_{a=1}^{3} \hat{C}_{a,v}^{\dagger} \hat{C}_{a,v}$$

  - Gaussian dust

$$\hat{H}_v := \frac{1}{2} \left[\hat{C}_v^{\dagger} + \hat{C}_v \right]$$

- Coherent state expectation value

$$\langle \mathbf{H} \rangle = \langle \Psi_Z^{\hbar} | \hat{\mathbf{H}} | \Psi_Z^{\hbar} \rangle = \mathbf{H}(Z, \bar{Z}) + \mathcal{O}(\hbar)$$

Giesel and Thiemann 06, 07, Thiemann 20,

Coherent State Path Integral and EoM

- Transition amplitude of gauge invariant coherent states (labeled by gauge orbit $[Z]$, $[Z']$)

$$A_{[Z],[Z']} = \langle \Psi^t_{[Z]}| \exp\left[\frac{i}{\hbar} T \hat{\mathbf{H}}\right] |\Psi^t_{[Z']}\rangle$$

Han,HL,19

- Discretize and insert N+1 overcompleteness relations ($u \in SU(2) \times G$)

$$\begin{aligned} A_{[Z],[Z']} &= \int \mathrm{d}u \left\langle \psi^\hbar_Z \right| \left[e^{\frac{i}{\hbar}\delta\tau \hat{\mathbf{H}}} \right]^N |\psi^\hbar_{Z'u}\rangle, \\ &= \int \mathrm{d}u \prod_{i=1}^{N+1} \mathrm{d}Z_i \, \langle \psi^\hbar_Z | \tilde{\psi}^\hbar_{Z_{N+1}} \rangle \langle \tilde{\psi}^\hbar_{Z_{N+1}} | e^{\frac{i\delta\tau}{\hbar}\hat{\mathbf{H}}} | \tilde{\psi}^\hbar_{Z_N} \rangle \langle \tilde{\psi}^\hbar_{Z_N} | e^{\frac{i\delta\tau}{\hbar}\hat{\mathbf{H}}} | \tilde{\psi}^\hbar_{Z_{N-1}} \rangle \cdots \\ & \quad \cdots \langle \tilde{\psi}^\hbar_{Z_2} | e^{\frac{i\delta\tau}{\hbar}\hat{\mathbf{H}}} | \tilde{\psi}^\hbar_{Z_1} \rangle \langle \tilde{\psi}^\hbar_{Z_1} | \psi^\hbar_{Z'u} \rangle \end{aligned}$$

- A path integral formula:

$$A_{[Z],[Z']} = \left\| \psi^\hbar_Z \right\| \left\| \psi^\hbar_{Z'} \right\| \int \mathrm{d}u \prod_{i=1}^{N+1} \mathrm{d}Z_i \, \nu[Z] \, e^{S[Z,u]/\hbar}$$

$\tilde{\varepsilon}_{i+1,i}\left(\frac{\delta\tau}{\hbar}\right)$ higher order terms in $\mathcal{O}(\delta\tau)$

- "effective action" $S[g,h]$ up to $O(\delta\tau)$ is given by

$$S[Z,u] = \sum_{i=0}^{N+1} \mathcal{K}\left(Z_{i+1}, Z_i\right) + i \sum_{i=1}^{N} \delta\tau \left[\frac{\langle \psi^\hbar_{Z_{i+1}} | \hat{\mathbf{H}} | \psi^\hbar_{Z_i} \rangle}{\langle \psi^\hbar_{Z_{i+1}} | \psi^\hbar_{Z_i} \rangle} + i \tilde{\varepsilon}_{i+1,i}\left(\frac{\delta\tau}{\hbar}\right) \right],$$

with

$$\mathcal{K}\left(Z_{i+1}, Z_i\right) = \frac{1}{\kappa} \sum_{e \in E(\gamma)} K(g_{i+1}, g_i) + \sum_{e \in E(\gamma)} \frac{1}{2Q^2} \underline{K}(\underline{g}_{i+1}, \underline{g}_i) + \sum_{v \in V(\gamma)} \left[\bar{z}_{i+1}(v) z_i(v) - \frac{1}{2} \bar{z}_{i+1}(v) z_{i+1}(v) - \frac{1}{2} \bar{z}_i(v) z_i(v) \right]$$
$$+ \sum_{v \in V(\gamma)} \left[\bar{\zeta}^\dagger_{i+1}(v) \bar{\zeta}_i(v) - \frac{1}{2} \bar{\zeta}^\dagger_{i+1}(v) \bar{\zeta}_{i+1}(v) - \frac{1}{2} \bar{\zeta}^\dagger_i(v) \bar{\zeta}_i(v) \right]$$

Lattice field theory

- Discrete Path integral

$$A_{[Z],[Z']} = \left\|\psi^{\hbar}_{Z}\right\| \left\|\psi^{\hbar}_{Z'}\right\| \int \mathrm{d}u \prod_{i=1}^{N+1} \mathrm{d}Z_i \, f(Z,u)\, e^{S^0[Z,u]/\hbar}, \qquad f(Z,u) = \nu[Z] e^{S^{\hbar}[Z,u]}$$

 where $S[Z,u] = \sum_{i=0}^{\infty} \hbar^i S^i[Z,u] =: S^0[Z,u] + S^{\hbar}[Z,u]$. $S^0[Z,u]$ is the semi-classical limit of S and $S^{\hbar}$ contains all quantum corrections (ignore $\mathcal{O}(\delta\tau^2)$ terms).

- $\hbar \ll 1$, Hormander's theorem applicable: Hormander,83, Theorem 7.7.5

$$\left| \int_K f(x) e^{iS^0(x)/\hbar} dx - e^{iS^0(x_0)/\hbar} \left[\det\left(\frac{S^{0\prime\prime}(x_0)}{2\pi i\hbar}\right)\right]^{-\frac{1}{2}} \sum_{s=0}^{k-1} (\hbar)^s L_s f(x_0) \right| \leq C(\hbar)^k \sum_{|\alpha|\leq 2k} \sup |D^\alpha f|$$

  - Amplitude is dominated by critical points satisfying semiclassical EoMs: $\delta_Z S(Z,u) = \delta_u S(Z,u) = 0$
  - Expansion of S to get S^0 requires semiclassical expansion of volume operator:

$$\hat{V}_v^{4q} = \langle \hat{Q}_v \rangle^{2q} \left[1 + \sum_{n=1}^{2k+1} (-1)^{n+1} \frac{q(1-q)\cdots(n-1+q)}{n!} \left(\frac{\hat{Q}_v^2}{\langle \hat{Q}_v \rangle^2} - 1\right)^n\right] + O(\hbar^{k+1}),$$

  one have $\langle \hat{Q}^N \rangle = \langle \hat{Q}_v \rangle^N [1 + \frac{3l_p^2}{8p^2} N(N-1)]$ in cosmology. Giesel and Thiemann 06, Dapor and Liegener 17

  - We need $\hat{V} \gg l_p^3$ in semi-classical limit, otherwise semi-classical expansion breaks down

The variation respect to $Z_i = (g_i, \underline{g}_i, z_i^r, \bar{z}_i^r, \zeta_{A,\nu,i})$ gives

	For $i = 1, \cdots, N$, at every $v \in V(\gamma)$	For $i = 2, \cdots, N+1$, at every $v \in V(\gamma)$
GR: $g_i(e) \to g_i(e) e^{\varepsilon_i^a(e)\tau^a}$	$\frac{\partial\left(K\left(Z_{i+1}, Z_i\right) + K\left(Z_i, Z_{i-1}\right)\right)}{\delta\tau\epsilon_i(v)} = -\frac{i\kappa}{a^2}\frac{\partial}{\partial\epsilon_i(v)}\frac{\langle\psi^{\hbar}_{Z_{i+1}}\|\hat{\mathbf{H}}\|\psi^{\hbar}_{Z_i}\rangle}{\langle\psi^{\hbar}_{Z_{i+1}} \mid \psi^{\hbar}_{Z_i}\rangle}$	$-\frac{\partial\left(K\left(Z_i, Z_{i-1}\right) + K\left(Z_{i-1}, Z_{i-2}\right)\right)}{\delta\tau\bar{\epsilon}_i(v)} = \frac{i\kappa}{a^2}\frac{\partial}{\partial\bar{\epsilon}_i(v)}\frac{\langle\psi^{\hbar}_{Z_i}\|\hat{\mathbf{H}}\|\psi^{\hbar}_{Z_{i-1}}\rangle}{\langle\psi^{\hbar}_{Z_i} \mid \psi^{\hbar}_{Z_{i-1}}\rangle}$
YM: $\underline{g}_i(e) \to \underline{g}_i(e) e^{\underline{\varepsilon}_i^a(e)\underline{T}^a}$	$\frac{\partial\left(\underline{K}\left(Z_{i+1}, Z_i\right) + \underline{K}\left(Z_i, Z_{i-1}\right)\right)}{\delta\tau\underline{\epsilon}_i(v)} = -2iQ^2\frac{\partial}{\partial\underline{\epsilon}_i(v)}\frac{\langle\psi^{\hbar}_{Z_{i+1}}\|\hat{\mathbf{H}}\|\psi^{\hbar}_{Z_i}\rangle}{\langle\psi^{\hbar}_{Z_{i+1}} \mid \psi^{\hbar}_{Z_i}\rangle}$	$-\frac{\partial\left(\underline{K}\left(Z_i, Z_{i-1}\right) + \underline{K}\left(Z_{i-1}, Z_{i-1}\right)\right)}{\delta\tau\bar{\underline{\epsilon}}_i(v)} = 2iQ^2\frac{\partial}{\partial\bar{\underline{\epsilon}}_i(v)}\frac{\langle\psi^{\hbar}_{Z_i}\|\hat{\mathbf{H}}\|\psi^{\hbar}_{Z_{i-1}}\rangle}{\langle\psi^{\hbar}_{Z_i} \mid \psi^{\hbar}_{Z_{i-1}}\rangle}$
Scalar: $z_i^r(v), \bar{z}_i^r(v)$	$\frac{\left[\bar{z}_{i+1}^r(v) - \bar{z}_i^r(v)\right]}{\delta\tau} = -i\frac{\partial}{\partial z_i^r(v)}\frac{\langle\psi^{\hbar}_{Z_{i+1}}\|\hat{\mathbf{H}}\|\psi^{\hbar}_{Z_i}\rangle}{\langle\psi^{\hbar}_{Z_{i+1}} \mid \psi^{\hbar}_{Z_i}\rangle}$	$\frac{\left[z_i^r(v) - z_{i-1}^r(v)\right]}{\delta\tau} = i\frac{\partial}{\partial\bar{z}_i^r(v)}\frac{\langle\psi^{\hbar}_{Z_i}\|\hat{\mathbf{H}}\|\psi^{\hbar}_{Z_{i-1}}\rangle}{\langle\psi^{\hbar}_{Z_i} \mid \psi^{\hbar}_{Z_{i-1}}\rangle}$
Fermions: $\zeta_{A,\nu,i}(v), \bar{\zeta}_{A,\nu,i}(v)$	$\frac{\left[\zeta_{A,\nu,i+1}(v) - \zeta_{A,\nu,i}(v)\right]}{\delta\tau} = i\frac{\partial}{\partial\bar{\zeta}_{A,\nu,i}(v)}\frac{\langle\psi^{\hbar}_{Z_{i+1}}\|\hat{\mathbf{H}}\|\psi^{\hbar}_{Z_i}\rangle}{\langle\psi^{\hbar}_{Z_{i+1}} \mid \psi^{\hbar}_{Z_i}\rangle}$	$\frac{\left[\bar{\zeta}_{A,\nu,i+1}(v) - \bar{\zeta}_{A,\nu,i}(v)\right]}{\delta\tau} = i\frac{\partial}{\partial\zeta_{A,\nu,i}(v)}\frac{\langle\psi^{\hbar}_{Z_i}\|\hat{\mathbf{H}}\|\psi^{\hbar}_{Z_{i-1}}\rangle}{\langle\psi^{\hbar}_{Z_i} \mid \psi^{\hbar}_{Z_{i-1}}\rangle}$

The variation with respect to $u \in SU(2) \times G$ leads to the closure condition for inital data

$$0 = \sum_{e, s(e)=v} p_1^a(e) - \sum_{e, t(e)=v} \Lambda^a_{\ b}\left(\vec{\theta}_1(e)\right) p_1^b(e) - \frac{i\kappa}{2a^2}\bar{\zeta}^\dagger(v)\frac{\tau^a}{2}\bar{\zeta}(v)$$

$$0 = \frac{1}{Q^2}\left(\sum_{e, s(e)=v} \underline{p}_1^I(e) - \sum_{e, t(e)=v} \underline{\Lambda}^I_{\ J}\left(\vec{\underline{\theta}}_1(e)\right) \underline{p}_1^J(e)\right) - \bar{\zeta}^\dagger(v) T^I_{R_f}\bar{\zeta}(v) - \bar{z}(v) T^I_{R_s}\bar{z}(v)$$

Semi-classical EoM

	For $i=1,\cdots,N$, at every $v\in V(\gamma)$	For $i=2,\cdots,N+1$, at every $v\in V(\gamma)$
GR: $g_i(e)\to g_i(e)e^{\varepsilon_i^a(e)\tau^a}$	$\frac{\partial\left(K\left(Z_{i+1},Z_i\right)+K\left(Z_i,Z_{i-1}\right)\right)}{\delta\tau\epsilon_i(v)}=-\frac{i\kappa}{a^2}\frac{\partial}{\partial\epsilon_i(v)}\frac{\langle\psi^\hbar_{Z_{i+1}}\lvert\hat{\mathbf{H}}\rvert\psi^\hbar_{Z_i}\rangle}{\langle\psi^\hbar_{Z_{i+1}}\mid\psi^\hbar_{Z_i}\rangle}$	$-\frac{\partial\left(K\left(Z_i,Z_{i-1}\right)+K\left(Z_{i-1},Z_{i-2}\right)\right)}{\delta\tau\bar\epsilon_i(v)}=\frac{i\kappa}{a^2}\frac{\partial}{\partial\bar\epsilon_i(v)}\frac{\langle\psi^\hbar_{Z_i}\lvert\hat{\mathbf{H}}\rvert\psi^\hbar_{Z_{i-1}}\rangle}{\langle\psi^\hbar_{Z_i}\mid\psi^\hbar_{Z_{i-1}}\rangle}$
YM: $\underline{g}_i(e)\to\underline{g}_i(e)e^{\underline{\varepsilon}_i^a(e)\underline{T}^a}$	$\frac{\partial\left(\underline{K}\left(Z_{i+1},Z_i\right)+\underline{K}\left(Z_i,Z_{i-1}\right)\right)}{\delta\tau\underline{\epsilon}_i(v)}=-2iQ^2\frac{\partial}{\partial\underline{\epsilon}_i(v)}\frac{\langle\psi^\hbar_{Z_{i+1}}\lvert\hat{\mathbf{H}}\rvert\psi^\hbar_{Z_i}\rangle}{\langle\psi^\hbar_{Z_{i+1}}\mid\psi^\hbar_{Z_i}\rangle}$	$-\frac{\partial\left(\underline{K}\left(Z_i,Z_{i-1}\right)+\underline{K}\left(Z_{i-1},Z_{i-1}\right)\right)}{\delta\tau\bar{\underline{\epsilon}}_i(v)}=2iQ^2\frac{\partial}{\partial\bar{\underline{\epsilon}}_i(v)}\frac{\langle\psi^\hbar_{Z_i}\lvert\hat{\mathbf{H}}\rvert\psi^\hbar_{Z_{i-1}}\rangle}{\langle\psi^\hbar_{Z_i}\mid\psi^\hbar_{Z_{i-1}}\rangle}$
Scalar: $z_i^r(v),\bar z_i^r(v)$	$\frac{\left[\bar z^r_{i+1}(v)-\bar z^r_i(v)\right]}{\delta\tau}=-i\frac{\partial}{\partial z^r_i(v)}\frac{\langle\psi^\hbar_{Z_{i+1}}\lvert\hat{\mathbf{H}}\rvert\psi^\hbar_{Z_i}\rangle}{\langle\psi^\hbar_{Z_{i+1}}\mid\psi^\hbar_{Z_i}\rangle}$	$\frac{\left[z^r_i(v)-z^r_{i-1}(v)\right]}{\delta\tau}=i\frac{\partial}{\partial\bar z^r_i(v)}\frac{\langle\psi^\hbar_{Z_i}\lvert\hat{\mathbf{H}}\rvert\psi^\hbar_{Z_{i-1}}\rangle}{\langle\psi^\hbar_{Z_i}\mid\psi^\hbar_{Z_{i-1}}\rangle}$
Fermions: $\zeta_{A,\nu,i}(v),\bar\zeta_{A,\nu,i}(v)$	$\frac{\left[\zeta_{A,\nu,i+1}(v)-\zeta_{A,\nu,i}(v)\right]}{\delta\tau}=i\frac{\partial}{\partial\bar\zeta_{A,\nu,i}(v)}\frac{\langle\psi^\hbar_{Z_{i+1}}\lvert\hat{\mathbf{H}}\rvert\psi^\hbar_{Z_i}\rangle}{\langle\psi^\hbar_{Z_{i+1}}\mid\psi^\hbar_{Z_i}\rangle}$	$\frac{\left[\bar\zeta_{A,\nu,i+1}(v)-\bar\zeta_{A,\nu,i}(v)\right]}{\delta\tau}=i\frac{\partial}{\partial\zeta_{A,\nu,i}(v)}\frac{\langle\psi^\hbar_{Z_i}\lvert\hat{\mathbf{H}}\rvert\psi^\hbar_{Z_{i-1}}\rangle}{\langle\psi^\hbar_{Z_i}\mid\psi^\hbar_{Z_{i-1}}\rangle}$

- Right hand side $\delta_{f_i}\lvert\psi^\hbar_{Z_i}\rangle\sim\hat O_{f_i}\lvert\psi^\hbar_{Z_i}\rangle$: $(f_i,\hat O_{f_i})=[(\epsilon_i^a,-\hat L^a),(\underline{\epsilon}_i^I,-\hat L^I),(z_i^r,\frac{1}{\sqrt{\hbar}}\mathfrak{a}_r^\dagger),(\bar\zeta_{A,\nu},-\frac{1}{\hbar}\hat\theta_{A,\nu}]$

$$\frac{\partial}{\partial f_i(v)}\frac{\langle\psi^\hbar_{Z_{i+1}}\lvert\hat{\mathbf{H}}\rvert\psi^\hbar_{Z_i}\rangle}{\langle\psi^\hbar_{Z_{i+1}}\mid\psi^\hbar_{Z_i}\rangle}=\frac{\langle\psi^\hbar_{Z_{i+1}}\lvert\hat{\mathbf{H}}\,\hat O_f(v)\rvert\psi^\hbar_{Z_i}\rangle\langle\psi^\hbar_{Z_{i+1}}\lvert\psi^\hbar_{Z_i}\rangle-\langle\psi^\hbar_{Z_{i+1}}\lvert\hat{\mathbf{H}}\rvert\psi^\hbar_{Z_i}\rangle\langle\psi^\hbar_{Z_{i+1}}\lvert\hat O_f(v)\rvert\psi^\hbar_{Z_i}\rangle}{\langle\psi^\hbar_{Z_{i+1}}\lvert\psi^\hbar_{Z_i}\rangle^2}$$

Always finite for arbitrarily small $\delta\tau$

- Left hand side must admit approximations $Z_i\to Z(\tau)$ which are differentiable in τ to be finite for arbitrary small $\delta\tau$. Otherwise no solution!

Therefore for all solutions, we can take the time continuous limit

Continuum time EoM

- Taking the time-continuum limit $\delta\tau \to 0$:
  - Left hand sides $\dfrac{\partial\left(\mathcal{K}\left(Z_{i+1}, Z_i\right) + \mathcal{K}\left(Z_i, Z_{i-1}\right)\right)}{\delta\tau\partial Z_i(v)}$ become time derivative.
  - Matrix elements of $\hat{\mathbf{H}}$ (hard to compute) are reduced to expectation values of $\hat{\mathbf{H}}$ (easier to compute). We have $\langle\hat{\mathbf{H}}\rangle = \mathbf{H} + \mathcal{O}(\hbar)$
  - Continuum EoMs
    - Gravity sector: $\begin{pmatrix} \dfrac{\mathrm{d}\boldsymbol{p}(e)}{\mathrm{d}\tau} \\ \dfrac{\mathrm{d}\boldsymbol{\theta}(e)}{\mathrm{d}\tau} \end{pmatrix} = P\left(\boldsymbol{p}, \boldsymbol{\theta}\right) \begin{pmatrix} \dfrac{\partial}{\partial \boldsymbol{p}(e)} \langle \tilde{\psi}^\hbar_Z | \hat{\mathbf{H}} | \tilde{\psi}^\hbar_Z \rangle \\ \dfrac{\partial}{\partial \boldsymbol{\theta}(e)} \langle \tilde{\psi}^\hbar_Z | \hat{\mathbf{H}} | \tilde{\psi}^\hbar_Z \rangle \end{pmatrix}$ Han, HL, 19,20
    - YM sector: $\begin{pmatrix} \dfrac{\mathrm{d}\underline{\boldsymbol{p}}(e)}{\mathrm{d}\tau} \\ \dfrac{\mathrm{d}\underline{\boldsymbol{\theta}}(e)}{\mathrm{d}\tau} \end{pmatrix} = \underline{P}\left(\underline{\boldsymbol{p}}, \underline{\boldsymbol{\theta}}\right) \begin{pmatrix} \dfrac{\partial}{\partial \underline{\boldsymbol{p}}(e)} \langle \tilde{\psi}^\hbar_Z | \hat{\mathbf{H}} | \tilde{\psi}^\hbar_Z \rangle \\ \dfrac{\partial}{\partial \underline{\boldsymbol{\theta}}(e)} \langle \tilde{\psi}^\hbar_Z | \hat{\mathbf{H}} | \tilde{\psi}^\hbar_Z \rangle \end{pmatrix}$
    - Scalar sector: $\dfrac{\mathrm{d}\phi^r(v)}{\mathrm{d}\tau} = \dfrac{\partial}{\partial \pi^r(v)} \langle \tilde{\psi}^\hbar_Z | \hat{\mathbf{H}} | \tilde{\psi}^\hbar_Z \rangle, \quad \dfrac{\mathrm{d}\pi^r(v)}{\mathrm{d}\tau} = -\dfrac{\partial}{\partial \phi^r(v)} \langle \tilde{\psi}^\hbar_Z | \hat{\mathbf{H}} | \tilde{\psi}^\hbar_Z \rangle$ Han, HL, 21
    - Fermions sector: $\dfrac{\mathrm{d}\zeta_{A,\nu}(v)}{\mathrm{d}\tau} = i\dfrac{\partial}{\partial \bar{\zeta}_{A,\nu}(v)} \langle \tilde{\psi}^\hbar_Z | \hat{\mathbf{H}} | \tilde{\psi}^\hbar_Z \rangle, \quad \dfrac{\mathrm{d}\bar{\zeta}_{A,\nu}(v)}{\mathrm{d}\tau} = i\dfrac{\partial}{\partial \zeta_{A,\nu}(v)} \langle \tilde{\psi}^\hbar_Z | \hat{\mathbf{H}} | \tilde{\psi}^\hbar_Z \rangle$
- Effective EoMs is the Hamiltonian flow generated by the classical discrete Hamiltonian:

$$\frac{\mathrm{d}Z(v)}{d\tau} = \{\mathbf{H}, Z(v)\}$$

- Dynamics is uniquely determined by the initial value as integration of ODE systems (with discretization on the lattice)
- Amplitude $A_{[Z],[Z']}$ will be exponentially suppressed in asymptotic analysis if $[Z]$ and $[Z']$ are not connected by the Effective EoMs (Classical forbidden regime):

  Complex critical points

Explorer Dynamics: example

Considering single real scalar field as matter field + cosmological constant

- Homogeneous and isotropic ansatz in full LQG EoMs Dapor and Liegener, 17, Han and HL, 19, 21

$$\theta^a(e_i(v)) = \theta\delta_I^a = \mu\beta K_0\delta_i^a, \qquad p^a(e_i(v)) = p\delta_i^a = \frac{2\mu^2}{\beta a^2}P_0\delta_i^a,$$

$$\phi(v) = \phi = \phi_0, \qquad \pi(v) = \pi = \mu^3\pi_0.$$

- μ_0-scheme effective dynamics with inflation: (suppose $P_0 > 0$)

$$\begin{aligned}\frac{4\beta^2\left[-2\mu^2\sqrt{P_0}\dot{K}_0 + \sin^4(\beta\mu K_0) + \Lambda\mu^2 P_0\right] - \sin^2(2\beta\mu K_0)}{\sqrt{P_0}} &= \kappa\beta^2\mu^2\sqrt{P_0}\left(\pi_0^2 P_0^{-3} - U\right),\\ \sqrt{P_0}\left[2\beta^2\sin(2\beta\mu K_0) - \left(\beta^2+1\right)\sin(4\beta\mu K_0)\right] + 2\beta\mu\dot{P}_0 &= 0,\end{aligned}$$

$$P_0^{3/2}\dot{\phi}_0 - \pi_0 = 0, \qquad P_0{}^{3/2}U'(\phi_0) = -2\dot{\pi}_0.$$

- Effective Physical Hamiltonian:

$$\begin{aligned}\frac{\mathbf{H}}{|V(\gamma)|} = C_v \quad = \quad & -\mu^3\left(\frac{3}{\beta^2\kappa\mu^2}P_0^{1/2}\sin^2(\beta\mu K_0)\left[-\beta^2 + \left(\beta^2+1\right)\cos(2\beta\mu K_0) + 1\right]\right.\\ & \left. -\frac{1}{2\kappa}P_0^{3/2}(4\Lambda + \kappa U(\phi_0)) - \frac{\pi_0^2}{2P_0^{3/2}}\right).\end{aligned}$$

- $U(\phi_0)$ Starobinsky inflationary potential

$$U(\phi_0) = \frac{3m^2}{\kappa}\left[1 - \exp\left(-\sqrt{\frac{\kappa}{3}}\phi_0\right)\right]^2$$

Problems with μ_0 Scheme

We have μ_0 scheme effective dynamics with inflation! However this introduce problems, due to the following requirement

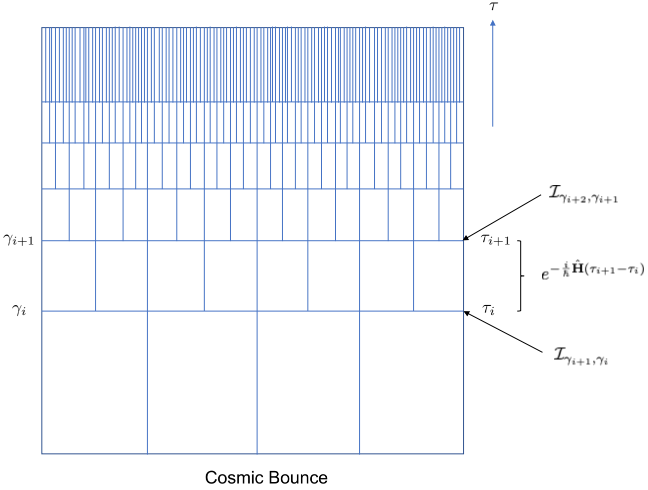
- $\theta = \beta K_0 \mu$ has to be sufficiently small at late time
  in order to approximate the classical theory on the continuum, s.t. $\sin\theta \sim \theta$
- μ can not be too small, otherwise Q will be to small thus breaks the semi-classical limit!

At the end of inflation period T,
$P_0 = e^{2\chi T}\mathring{P}_0$ and $K_0 = \chi e^{\chi T}\mathring{P}_0^{1/2}$, P_0, K_0 late time, $\mathring{P}_0$ early time
if we set $e^{\chi T} \sim 10^{24}$ and $\chi \sim 10^{-6} l_P^{-1}$, $\sin(\beta\mu K_0) \simeq \beta\mu K_0$ requires $\beta\mathring{p} \sim 2\mu^2 \mathring{P}_0 = 10^{-20} l_p^2$,
too small!

How to make μ^2 scales with P_0: similar to $\bar{\mu}$-scheme?

A possible resolution: Dynamical Lattice refinement

Refine the spatial lattice during the time evolution: spacetime lattice, like spinfoam, possible since we start with discrete path integral.



- $\mathcal{I}_{\gamma_i,\gamma_{i-1}} : \mathcal{H}_{\gamma_{i-1}} \to \mathcal{H}_{\gamma_i}$
- $\mathcal{I}_{\gamma_i,\gamma_{i-1}}$: preserves the homogeneity and isotropy, and semi-classical limit of expectation values: P_0, K_0, ϕ_0, π_0.
- Possible definition of $\mathcal{I}_{\gamma_i,\gamma_{i-1}}$ – on Fourier space by identifying infrared mode

Time continuum limit $\delta\tau \to 0$ makes μ a smooth function of time $\mu_i \to \mu(\tau)$:

Map from function $\mu(\tau)$ to solution space

Two possible scheme: μ_{min}-scheme and ensemble average scheme

- μ_{min}-scheme:
  choice of μ to minimize the discreteness while still validating the semiclassical volume expansion:

  UV cut-off Δ (a small area scale) such that $V > \Delta^{3/2} \gg \ell_P^3$

  μ_{min} is chosen to saturate this UV cut-off

$$\mu_{min}(\tau) = \sqrt{\frac{\Delta}{P_0[\mu_{min}](\tau)}}$$

  such $\mu_{min}(\tau)$ is uniquely defined according to the EoMs.

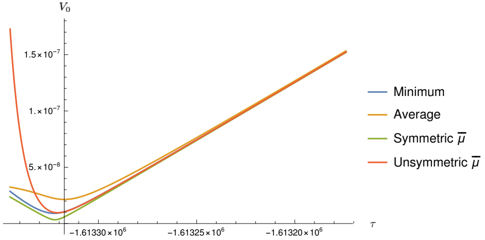
- Ensemble average scheme:
  Ensembles of different lattice $\mathfrak{F}(\tau)$ with certain probability distribution $\mathfrak{B}(\tau)$, and ensemble average.
  $\mathfrak{B}(\tau)$ determined by all possible sub-lattices from the most-refined lattice at time τ_0.

$$\overline{O} = \int_{\{\mu\}} D\mu\, \mathfrak{P}(\mu) O[\mu] \sim O[\overline{\mu(\tau)}], \quad \overline{\mu(\tau)} = 2\mu_{min}$$

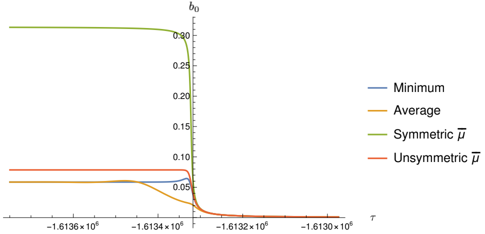
- Scaling invariance is recovered.

$$P_0[\mu_{min}](\tau) \to \alpha P_0[\mu_{min}](\tau), \quad \pi_0[\mu_{min}](\tau) \to \alpha^{3/2} \pi_0[\mu_{min}](\tau),$$
$$b_0[\mu_{min}](\tau) \to b_0[\mu_{min}](\tau), \quad \phi_0[\mu_{min}](\tau) \to \phi_0[\mu_{min}](\tau).$$

similar for the other scheme


V_0
1.5×10^{-7}
$1.\times10^{-7}$
$5.\times10^{-8}$
-1.61330×10^{6}
-1.61325×10^{6}
-1.61320×10^{6}
τ
Minimum
Average
Symmetric $\overline{\mu}$
Unsymmetric $\overline{\mu}$

(a)


b_0
0.30
0.25
0.20
0.15
0.10
0.05
-1.6136×10^{6}
-1.6134×10^{6}
-1.6132×10^{6}
-1.6130×10^{6}
τ
Minimum
Average
Symmetric $\overline{\mu}$
Unsymmetric $\overline{\mu}$

(b)

Effective Dynamics

	average	μ_{min}	unsymmetric $\bar{\mu}$	symmetric $\bar{\mu}$
Asymptotic FRW at late time	Yes	Yes	Yes	Yes
Singularity resolution and bounce	Yes	Yes	Yes	Yes
Critical density at the bounce	$\frac{3}{16\kappa\Delta\left(1.6+3\times10^{-4}\,\overline{\phi}_0(\tau_B)\sqrt{\Delta}\right)}$ (for $\beta=1$)	$\frac{3}{2\beta^2\left(\beta^2+1\right)\kappa\Delta}$	$\frac{3}{2\beta^2\left(\beta^2+1\right)\kappa\Delta}$	$\frac{16}{\beta^2\Delta\kappa}$
dS phase in the past to the bounce	Yes	Yes	Yes	No

Linear Cosmological Perturbations from Full Theory

- Linear perturbation of lattice variables $\theta^a(e), p^a(e), \phi(v), \pi(v)$:

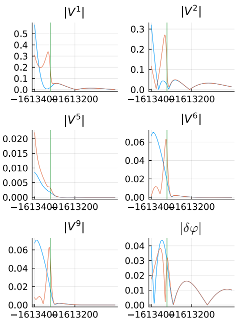
$$\theta^a\left(e_I(v)\right) = \mu\left[\beta K_0 \delta_I^a + \mathcal{X}^a\left(e_I(v)\right)\right], \qquad p^a\left(e_I(v)\right) = \frac{2\mu^2}{\beta a^2} P_0\left[\delta_I^a + \mathcal{Y}^a\left(e_I(v)\right)\right],$$

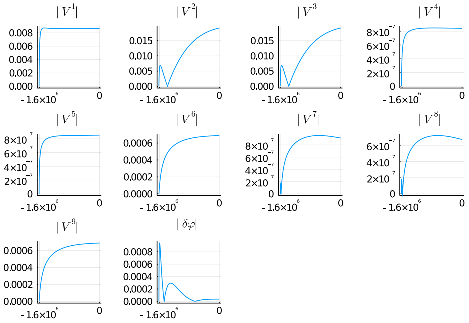
$$\phi(v) = \phi_0 + \delta\varphi(v), \qquad \pi(v) = \mu^3\left[\pi_0 + \delta\pi(v)\right]$$

- Lattice Fourier transform

$$V^\rho(\tau, v) = V^\rho(\tau, \vec{\sigma}) = \frac{1}{L^3} \sum_{\vec{k} \in \left(\frac{2\pi}{L}\mathbb{Z}\right)^3,\ |k^I| \le \frac{\pi}{\mu}} e^{i\vec{k}\cdot\vec{\sigma}} \tilde{V}^\rho(\tau, \vec{k}), \quad \sigma^I \in \mu\mathbb{Z}.$$

- Numerical evaluation of EoMs with background solutions in both scheme





Fermions Field

Massless fermions field on cosmological background:

Fermions are quadratic – no coupling between linear cosmological perturbations and fermions

- Effective EoMs (in lattice Fourier modes)

$$\frac{\mathrm{d}\theta^\dagger(\vec{k})}{\mathrm{d}\tau} = \frac{i\theta^\dagger(\vec{k})}{\mu\sqrt{P_0}}\left(\sum_{j=1,2,3}\left(\sin\left(\mu k^j\right)\sigma_j\cos(\beta\mu K_0) + \cos\left(\mu k^j\right)\sin\left(\frac{\beta\mu K_0}{2}\right)\right) + \beta\frac{\kappa_2\sin(2\beta\mu K_0)}{4}\right)$$

- In flat spacetime limit $K_0 \to 0, P_0 \to const$

$$\frac{\mathrm{d}\theta^\dagger(\vec{k})}{\mathrm{d}\tau} = \frac{i\theta^\dagger(\vec{k})}{\mu\sqrt{P_0}}\sum_{j=1,2,3}\left(\sin\left(\mu k^j\right)\sigma_j\right)$$

- 7 Doublers! (same propagator as in lattice field theory with time continuum limit):

$$D_F = \frac{1}{\left(k^0 + \sum_{j=1,2,3}\frac{\sin\left(\mu k_j\right)}{\mu}\sigma_j\right)}$$

7 extra pole except $p = 0$ due to periodicity of $\sin$ function at the Brillouin zone boundary

Fermions Field

Remove the doubler:

- Add Wilson terms/Ginsparg-Wilson fermions : explicit breaking of chiral symmetry

$$\mathcal{C}_F^W(v) = \mathcal{C}^F(v) - r\left(\widehat{\frac{\mathrm{sgn}(e)}{V}}\right)_v \frac{a^2\beta}{8}\sum_{s_1s_2s_3}\sum_j s_j\left[\sqrt{|\hat{\vec{X}}^j(v)|^2}\hat{\mathscr{W}}(e_{v;js_j})\right]$$

$$\hat{\mathscr{W}}(e) = \hat{\theta}^\dagger(v)\,\hat{h}(e)R_f\left(\underline{\hat{h}}(e)\right)\hat{\theta}(t(e)) + \hat{\theta}^\dagger(t(e))\,R_f\left(\underline{\hat{h}}(e)^{-1}\right)\hat{h}(e)^{-1}\hat{\theta}(v) - 2\hat{\theta}^\dagger(v)\,\hat{\theta}(v)$$

s.t. $$D_F = \frac{1}{\left(k^0 + \frac{r}{\mu}(1-\cos(\mu k^j)) + \sum_{j=1,2,3}\frac{\sin(\mu k^j)}{\mu}\sigma_j\right)}$$

- Ensemble average (implicit breaking of chiral symmetry?):
  Ensembles of different lattice $\{\mu\}$ with certain probability distribution $\mathfrak{P}(\mu)$, and ensemble average – working in progress

$$\overline{O} = \int_{\{\mu\}} D\mu\,\mathfrak{P}(\mu)O[\mu]$$

A package for semi-classical evaluation

- Fast analytic evaluation of semi-classical Hamiltonian with the help of SymPy and SymEngine:
  Some Benchmarks with general holonomy $h(e_{i,s})_{AB}$ and flux $p^a(e_{i,s})$ on single thread of Epyc 7742 at 3.4Ghz:
  - Euclidean Hamiltonian takes only seconds
  - Extrinsic curvature $K_v = \sum_{v'} \{\mathcal{C}_0(v), V(v')\}$ takes around 1 min (with 54144 different terms as $f(p,h)$)
  - Thiemann's Lorentzian Hamiltonian 10 min (with millon's of different terms)
    Previous calculation takes several days with 40 cores on the same server
- Substitution of arbitrary ansatz for flux and connections possible. With very fast series expansion and multi-threads support
- Code generation and Julia interface with Differentialequations.jl (diffeq.sciml.ai) for numerical evaluation of the EoMs (as ODEs)
  examples: cosmology and cosmological perturbations

Conclusion

- We have present a path integral formulation of LQG transition amplitude with matter couplings (YM, Fermions and scalar fields)
- We compute its semi-classical limit and derive the effective equations of motion

  The semiclassical-continuum limit reproduces the classical gravity-dust-matter theory
- A framework for evaluation of semiclassical dynamics of full LQG with matter fields

  Python/Julia based, both analytical and numerical
- Extend single fixed lattice to ensembles of lattices: dynamical lattice refinement and ensemble average

  $\bar{\mu}$-scheme like dynamics in cosmology

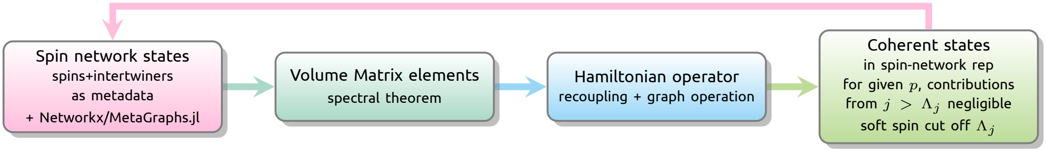
Outlook: full quantum evaluation: numerical package

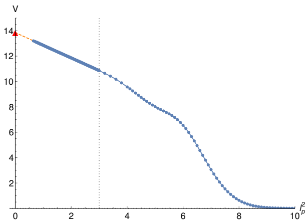
- Quantum regime where $Q \sim l_p^3$:
  semiclassical expansion of volume operator not applicable $Q \sim l_p^3$:

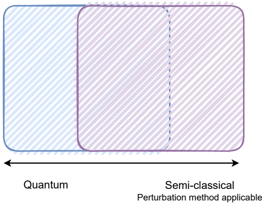
$$\hat{V}_v^{4q} = \langle \hat{Q}_v \rangle^{2q} \left[1 + \sum_{n=1}^{2k+1} (-1)^{n+1} \frac{q(1-q)\cdots(n-1+q)}{n!} \left(\frac{\hat{Q}_v^2}{\langle \hat{Q}_v \rangle^2} - 1 \right)^n \right] + O(\hbar^{k+1})$$

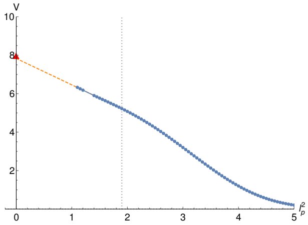
  Hard to keep track of all corrections and sub-dominate critical points on Lefschetz thimble

- A calculation framework: As general as possible (arbitrary valent spin-networks) —- working in progress









Thank you for your attention!

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