New applications for LQG

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Plan

- Introduction to LQG
- position / momentum representations
  L, Sahlmann - 2014 (coming soon)
  - Application 1: a new operator

$$\hat{q}^{ab}\hat{\phi}_{,a}\hat{\phi}_{,b}\mathrm{det}\hat{q}$$

  - Application 2: The BF vacuum
- Symmetric scalar constraint operators

$$\hat{C}(N)$$

for all the laps functions L, Sahlmann - 2014

Σ - an underlying 3d manifold, $x^a = x^1, x^2, x^3$ local coordinates

- The variables:
  - $\mathrm{su}(2)$ - the Lie algebra of SU(2), $\tau_i = \tau_1, \tau_2, \tau_3 \in \mathrm{su}(2)$
  - $(A^i_a,\ E^a_i)$ $a, b = 1, 2, 3, i = 1, 2, 3$ - the field variables
  - $\{A^i_a(x), E^b_j(y)\} = \delta^i_j \delta^b_a \delta(x, y)$
  - $\{A, A\} = 0 = \{E, E\}$

- The relation with intrinsic/extrinsic geometry e^i_a / K_{ab} of Σ
  Ashtekar, Barbero, Immirzi

$$-2\mathrm{Tr}(\tau_i \tau_j) = \delta_{ij},$$

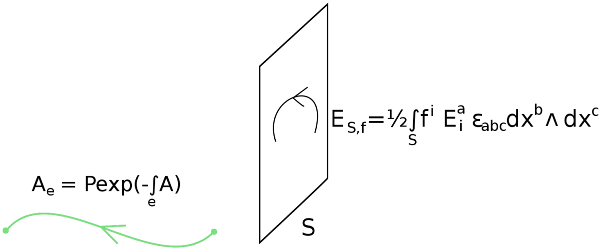
$$A^i_a = \Gamma^i_a + \gamma K^i_a, \quad E^a_i = \frac{1}{16\pi G\gamma} e^j_b e^k_c \epsilon^{abc} \epsilon_{ijk}.$$

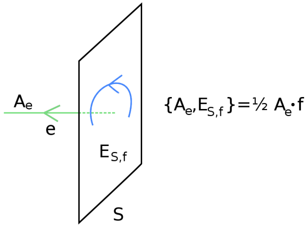
- Other fields

$$\{\phi_\alpha(x), \pi^\beta(y)\} \;=\; \delta_\alpha^\beta \delta(x,y)$$
$$\{\phi, \phi\} \;=\; 0 \;=\; \{\pi, \pi\}$$

- The constraints (I - class)
  - $\int_\Sigma d^3x N(x) C(x) =: C(N) \;=\; C_{\rm gr}(N) \;+\; C_{\rm matt}(N)$
  - $\int_\Sigma d^3x N^a(x) C_a(x) \;=:\; C(\vec{N}) \;=\; C_{\rm gr}(\vec{N}) \;+\; C_{\rm matt}(\vec{N})$
  - $\int_\Sigma d^3x \Lambda^i G_i(x) =: G(\Lambda) \;=\; G_{\rm gr}(\Lambda) \;+\; G_{\rm ferm}(\Lambda)$
  - Other constraints ($G_{\rm YM}(\Lambda_{\rm YM})$)
- The free functions:
  - $N \in C(\Sigma)$ - a laps function
  - $\vec{N} \in \Gamma(T(\Sigma))$ - a shift vector field
  - $\Lambda \in C(\Sigma, {\rm su}(2))$.

Rovelli, Smolin 1988





Ashtekar, L 1992

$$e : [t_0, t_1] \to \Sigma$$

$$h_e(A) := \mathrm{Pexp} \int_e -A$$

$$\Psi(A) = \psi(h_{e_1}(A), ..., h_{e_n}(A)), \qquad \psi \in \mathrm{Poly}(\mathrm{SU}(2)^n) \qquad (1)$$

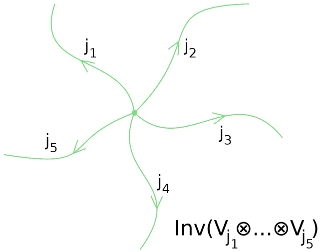
$$\mathrm{Cyl} := \{\Psi \in C(\mathcal{A}) : (1),\ \{e_1, ..., e_n\} \ \text{embedded graph in } \Sigma\}$$

$$\int d\mu(A)\Psi(A) = \int dg_1...dg_n\psi(g_1, ...g_n)$$

$$\mathcal{H}_{\mathrm{gr}} = \mathrm{L}^2(\Omega^{(1)}(\Sigma) \otimes \mathrm{su}(2),\ \mu)$$

$$U_g\Psi(A) = \Psi(g^{-1}Ag + g^{-1}dg) \qquad (2)$$

The invariant elements in $\mathcal{H}_{\rm gr}$ *Penrose 1970(?), Rovelli, Smolin, Baez 1993*:



$$\Gamma \ni e_I \ \mapsto j_I, \quad \Gamma \ni v \ \mapsto \ \iota_v \ \in \ {\rm Inv}\, V_{j_1} \otimes \ \ldots \ \otimes V_{j_k}$$
$$|\Gamma, j, \iota > \text{ - a spin-network state}$$

$$[\hat{q}, \hat{p}] = i\hbar \tag{3}$$

$$\hat{p}|p> = p|p>, \qquad e^{\frac{i}{\hbar}p'\hat{q}}|p> = |p+p'> \tag{4}$$

$$\hat{q}|q> = q|q>, \qquad e^{-\frac{i}{\hbar}q'\hat{p}}|q> = |q+q'> \tag{5}$$

$$|q>: p \mapsto e^{\frac{i}{\hbar}qp}, \qquad |p>: q \mapsto e^{-\frac{i}{\hbar}qp} \tag{6}$$

$$< q|p> = e^{\frac{i}{\hbar}qp}$$

$$\{\phi(x), \pi(y)\} = \delta(x,y), \qquad [\hat{\phi}(x), \hat{\pi}(y)] = i\hbar\delta(x,y) \quad (7)$$

The polymer representation:

$$\int d^3x \hat{\pi}(x)\varphi(x)|p> = \sum_{x\in\Sigma} p_x \varphi(x)|p> \quad (8)$$

$$p : \Sigma \to \mathbb{R}, \qquad |\mathrm{supp}\ p| < \infty \quad (9)$$

$$e^{\frac{i}{\hbar}\sum_{x\in\Sigma} p'_x \hat{\phi}(x)}|p> = |p+p'> \quad (10)$$

$$|p> : \ \phi \mapsto e^{\frac{i}{\hbar}\sum_{x\in\Sigma} p_x \phi(x)} \quad (11)$$

Define $<\varphi|$ to be:

$$<\varphi|p> := e^{\frac{i}{\hbar}\sum_{x\in\Sigma} p_x \varphi(x)} \quad (12)$$

$$< \varphi|p> := \ e^{\frac{i}{\hbar}\sum_{x\in\Sigma} p_x \varphi(x)} \tag{13}$$

$$\Rightarrow$$

$$< \varphi|e^{\frac{i}{\hbar}\sum_{x\in\Sigma} p_x \hat{\phi}(x)} \ = < \varphi|e^{\frac{i}{\hbar}\sum_{x\in\Sigma} p_x \varphi(x)} \tag{14}$$

$$< \varphi|e^{\frac{i}{\hbar}\int d^3x \varphi'(x)\hat{\pi}(x)} \ = < \varphi + \varphi'| \tag{15}$$

$$\Rightarrow$$

$$< \varphi|\hat{\phi}(x) \ = < \varphi|\varphi(x). \tag{16}$$

The dual Polymer Representation

$$< \varphi | \, \varphi' > \; = \; \delta_{\varphi,\varphi'} \; = 0 \text{ or } 1 \tag{17}$$

$$\mathcal{H}_\phi \; = \; \mathrm{Span}(|\varphi > : \; \varphi \in C(\Sigma)) \tag{18}$$

$$\hat{\phi}|\varphi > = \; \varphi(x)|\varphi >, \qquad e^{-\frac{i}{\hbar}\int d^3x \varphi'(x)\hat{\pi}(x)}|\varphi > = \; |\varphi + \varphi' > \tag{19}$$

Notice that

$$\hat{\phi}_{,a\ldots}(x)|\varphi > = \; \varphi_{,a\ldots}(x)|\varphi > . \tag{20}$$

L,Sahlmann 2014 (coming)

$$\mathcal{H}_\phi \otimes \mathcal{H}_{\rm gr} \ \ni |\varphi > \otimes |\Gamma, j, \iota >$$

$$\int d^3x N(x) \sqrt{\hat{\phi}_{,a} \hat{\phi}_{,b} \hat{E}^a_i \hat{E}^b_i} \ |\varphi > \otimes |\Gamma, j, \iota > =$$

$$= \ (8\pi G\gamma)^2 \left(\sum_I \sqrt{j_I(j_I+1)} \int_{e_I} N|d\varphi| \right) \ |\varphi > \otimes |\Gamma, j, \iota >$$

where

$$\int_e N|d\varphi| \ = \ \int dt N(e(t)) |\frac{d\varphi(t)}{dt}|$$

Similar results in the LQG literature:

- the Ma-Ling operator Ma,Ling 2000 defined in $\mathcal{H}_{\rm gr}$

$$\hat{Q}(\omega)|\Gamma, j, \iota> \ = \int d^3x \sqrt{\hat{\omega}_a \hat{\omega}_b \hat{E}^a_i \hat{E}^b_i}\, |\Gamma, j, \iota> = \tag{21}$$

$$(8\pi G\gamma)^2 \left(\sum_I \sqrt{j_I(j_I+1)} \int_{e_I} |\omega| \right) \ |\Gamma, j, \iota> \tag{22}$$

- The Schroedinger equation of the Rovelli-Vidotto QM on graphs Rovelli, Vidotto 2010

Toy equations

A toy model of the quantum scalar constraint

$$\int d^3x \left(\widehat{\pi}(x) \; + \; a\sqrt{\widehat{E}^a_i(x)\widehat{E}^b_i(x)\widehat{\phi}_{,a}(x)\widehat{\phi}_{,b}(x)} \right) \Psi \;=\; 0. \qquad (23)$$

A general (modulo linearity) solution is:

$$\Psi \;=\; e^{ia\int d^3x \widehat{\phi}\sqrt{\widehat{E}^a_i\widehat{E}^b_i\widehat{\phi}_{,a}\widehat{\phi}_{,b}}} \sum_{\varphi} \psi_0(\varphi) < \varphi| \otimes < \Gamma, j, \iota|$$

where $\varphi \mapsto \psi_0(\varphi)$, is an arbitrary function which satisfies the following condition

$$\frac{d}{d\epsilon}\psi_0(\varphi + \epsilon) \;=\; 0\;. \qquad (24)$$

$$[\hat{A}_a(x), \hat{E}^b(x)] \; = \; i\hbar\delta_a^b\delta(x,y) \tag{25}$$

The LQG momentum representation:

$$\int d^3x \hat{E}^a a_a|e> = \; \hbar\int_e a|e> \qquad\qquad e^{i\int_{e'}\hat{A}}|e> = \; |e\circ e'> \tag{26}$$

The dual position representation spanned by the dual states $< a|$:

$$< a|e> := \; e^{\frac{i}{\hbar}\int_e a}. \tag{27}$$

$$< a|a'> = \; \delta_{a,a'}$$

In this representation

$$< a|\hat{A}_a(x) \; = < a|a_a(x). \tag{28}$$

In particular

$$< a = 0| \; = \; \mathrm{BF} \quad \mathrm{vacuum}$$

proposed by Dittrich, Geiller 2014

The issue

- $\hat{C}_{\rm gr}(N)$ not defined in the kinematical $\mathcal{H}_{\rm gr}$
- $\hat{C}(N) : \mathcal{H}_{\rm Diff} \to$ suitable dual space , N breaks Diffs
- In principle we can write

$$\hat{C}_{\rm gr}(N)\Psi \;=\; 0$$

however:

- solutions non-normalizable
- to large space, which Ψ to select?
- $(\Psi|\Psi')_{\rm phys} =?$
- For GR coupled to a scalar field we need the Rovelli-Smolin

$$\int_\Sigma d^3x\sqrt{-2\sqrt{\det\hat{E}(x)}\hat{C}_{\rm gr}(x)}$$

how to define the $\sqrt{\hat{C}_{\rm gr}(x)}$?

Obstacles

- Perhaps, we can extend $\mathcal{H}_{\rm Diff} \subset \mathcal{H}'$ to accomodate $\hat{C}(N)$???
- The obstacle: Ψ Diff invariant

$$\Rightarrow \quad (\hat{C}(N)\Psi \mid \hat{C}(N)\Psi)' \;=\; 0$$

for every $N \in C_0^\infty(\Sigma)$. L, Marolf 1999

Partial ways out

- Take $N = 1$,

$$\hat{C}(1) : \mathcal{H}_{\rm Diff} \to \mathcal{H}_{\rm Diff}$$

$$\hat{C}_{\rm gr,sym} \ := \ \frac{1}{2}\left(\hat{C}_{\rm gr}(1) + \hat{C}^{\dagger}_{\rm gr}(1)\right).$$

  But $N = 1$ is not enough...

- Try to define directly
  - either

$$\int d^3x \sqrt{-\det \hat{E}(x)\hat{C}(x)}$$

  - or

$$\int d^3x \frac{\hat{C}^2(x)}{\sqrt{\det \hat{E}(x)}}$$

  the Master constraint program (Thiemann)

L, Sahlmann 2014

$$\mathcal{H}_{\rm new} = \bigoplus_{\{x_1,...,x_k\}\subset\Sigma} \mathcal{H}_{\{x_1,...,x_k\}}$$

$\mathcal{H}_{\{x_1,...,x_k\}}$ is spanned by all the $|\Gamma, j, \iota>$ based at $\{x_1, ..., x_k\}$, averaged with respect to $\mathrm{Diff}(\Sigma, \{x_1, ..., x_k\})$.

We define operators in $\mathcal{H}_{\mathrm{new}}$ by passing by the duality from the kinematical $\mathcal{H}_{\mathrm{gr}}$ operators of suitable symmetries, or we derive $\hat{C}_{\mathrm{gr}}(N)$ from scratch (Thiemann's regularization Thiemann 1997 works in $\mathcal{H}_{\mathrm{new}}$ very well, for Ricci see Assanioussi, Alesci, L 2013):

$$\mathcal{O}(x) \;=\; \sqrt{\det E(x)},\; \sqrt{\det E}(x)\mathrm{Ric}(x), \hat{C}_{\mathrm{gr}}(x).$$

We obtain operators of the following form

$$d^3xN(x)\widehat{\mathcal{O}}(x) \;=\; \sum_{x\in\Sigma} N(x)\hat{\mathcal{O}}_x$$

$$\hat{\mathcal{O}}_x : \mathcal{H}_{\{x_1,...,x_k\}} \to \mathcal{H}_{\{x_1,...,x_k\}}$$

$$\hat{\mathcal{O}}_x : \mathcal{H}_{\{x_1,...,x_k\}} \;\to\; 0 \;\; \text{unless} \;\; x = x_1, ..., x_k$$

The quantum scalar contraint

With this result we can:

- Consider $\hat{C}^\dagger(N)$ (it has sufficiently large domain).
- Define

$$\hat{C}(x)_{\rm sym} = \frac{1}{2}(\hat{C} + \hat{C}^\dagger)$$

- Find a self adjoint extension $\hat{C}_{\rm s.a.}$
- Spectrally expand each

$$\mathcal{H}_{\{x_1,\dots,x_k\}} = \int^{\oplus} dc_1 \dots dc_k \mathcal{H}^{c_1\dots c_k}_{\{x_1,\dots,x_k\}}$$

- promote $\mathcal{H}^{0\dots0}_{\{x_1,\dots,x_k\}}$ to be solutions to the quantum scalar constraint

Solutions to the matter free LQG

The elements of $\mathcal{H}_{\{x_1,...,x_k\}}$ have to be farther averaged with respect to the vertex non-preserving Diff(Σ) (the vector constraint),

$$\mathcal{H}_{\{x_1,...,x_k\}}\ \eta_{\rm new}(\Psi)\ \mapsto\ \frac{1}{k!}\sum_{[f]\in {\rm Diff}(\Sigma)/{\rm Diff}(\Sigma)_{x_1,...,x_k}}\eta_{\rm new}(U_f\Psi)$$

The map passes to the $\mathcal{H}^{0...0}_{\{x_1,...,x_k\}}$ space, and it's image defines a Hilbert space

$$\mathcal{H}_{(k)}$$

of k-vertex solutions to the scalar AND vector constraint.
The full Hilbert space is

$$\mathcal{H}_{\rm phys}\ =\ \oplus_k\mathcal{H}_{(k)}$$

How to use the new gadgets?

We apply it to (set $8\pi G\gamma = 1$)

$$\hat{\pi}(x) \ = \ \pm\sqrt{-\hat{\phi}_{,a}\hat{\phi}_{,b}\hat{E}^a_i\hat{E}^b_i - 2V(\hat{\phi(x)})|\det\hat{E}| - 2\sqrt{|\det\hat{E}|}\hat{C}_{\rm gr}}$$

Where, given operators

$$\int_\Sigma d^3x\sqrt{\hat{A}(x)}, \quad \int_\Sigma d^3x\sqrt{\hat{B}(x)}, \ \ldots$$

Using sufficiently fine partitions $\Sigma \ = \ \bigcup_k \Sigma_k$ we define

$$\int_\Sigma d^3x\sqrt{\hat{A}(x)+\hat{B}(x)+\ldots} \ \ |\varphi > \otimes|\Gamma, j, \iota >:=$$

$$\sum_k\sqrt{\left(\int_{\Sigma_k} d^3x\sqrt{\hat{A}(x)}\right)^2 + \left(\int_{\Sigma_k} d^3x\sqrt{\hat{B}(x)}\right)^2 + \ldots} \ \ |\varphi > \otimes|\Gamma, j, \iota >$$

provided the RHS is independent of the refinements.

Thank You