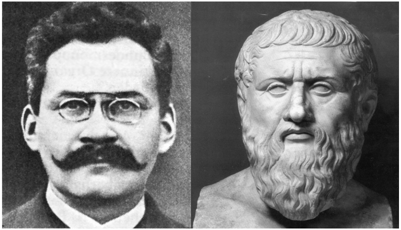
Shape Dynamics

International Loop Quantum Gravity Seminar

Tim A. Koslowski

Perimeter Institute for Theoretical Physics
Waterloo, Ontario



November 1^{st}, 2011

Literature

Original:

- H. Gomes, S. Gryb, T.K.: "Einstein gravity as a 3D conformally invariant theory," CQG 28 (2011) 045005. [arXiv:1010.248];
- H. Gomes, T.K.: "The Link between General Relativity and Shape Dynamics," [arXiv:1101.5974];
- H. Gomes, S. Gryb, T.K., F. Mercati: "The gravity/CFT correspondence," [arXiv:1105.0938];
- T. Budd, T.K. : "Shape Dynamics in 2+1 Dimensions," [arXiv:1107.1287];
- H. Gomes, T.K.: "Coupling Shape Dynamics to Matter Gives Spacetime," [arXiv:1110.3837];
- T.K.: "Loop Quantization of Shape Dynamics", (sorry, not yet out)

Introduction:

- H. Gomes: "The Dynamics of Shape," [arXiv:1108.4837];
- T.K.: "Shape Dynamics," [arXiv:1108.5224]

Background:

- J. Barbour: "Shape Dynamics: An Introduction," [arXiv:1105.0183]

Work in progress:

- T.K.: "Symmetry Doubling";
- H. Gomes, T.K.: "Constructing Shape Dynamics";
- H. Gomes, S. Gryb, T.K., N.N.: "Shape Dynamics Perturbation Theory."

Outline

- **Introduction**
- **Symmetry Trading:** Linking Theories, Canonical best Matching
- **Construction of Shape Dynamics:** Best matching ADM, Linking Theory, Shape Dyanmics
- **Challenges and Answers:** Spacetime Picture, Nonlocality, Non-CMC solutions
- **Tentative Loop Quantization:** SD in Ashtekar variables, kinematic loop quantization, tentative "dyanmics", "interpretation"
- **Summary and Outlook:** *"There are many open questions."*

What is Shape Dynamics?

Short Answer:

Shape Dynamics is a formulation of GR where refoliation symmetry is traded for local spatial conformal symmetry.

It is just a reformulation of GR. Why bother?

- **Ontology**: Different Symmetry ⇒ different theory space for effective field theory (Hořva)
- **Access to new aspects**: e.g. classical large CMC-volume/CFT correspondence
- **Quantization Opportunities**: e.g. Dirac quantization of metric gravity on 2+1 sphere and torus

Canonical Framework

Canonical System $(\Gamma,\{.,.\},H,\{\chi_i^{fir.}\}_{i\in\mathcal{I}},\{\chi_j^{sec.}\}_{j\in\mathcal{J}})$

- Obtained from singular Legendre transform of **consistent** Lagrangian
- form of constraints is **not** determined by Legendre transform
- assume regular, irreducible first class constraints $\{\chi_i^{fir.}\}_{i\in\mathcal{I}}$
- **Dirac conjecture:** First class secondary constraints generate gauge transformations. (counter example: $L=e^y\dot{x}^2$)
- **Gauge fixing:** regular, irreducible set $\{\sigma_i\}_{i\in\mathcal{I}}\cup\{\chi_i^{fir.}\}_{i\in\mathcal{I}}$, such that $\{\chi_i^{fir.},\sigma_j\}$ is invertible
- **Reduced phase space:** $\Gamma_{red}=\Gamma|_{\{\chi_j^{sec.}=0\}_{j\in\mathcal{J}}}$ with **Dirac** bracket $\{.,.\}_D$

From now on:

- assume energy conservation constraint $\chi_o^{fir.}=H-E$
- second class constraints are solved, i.e. $\{\chi_j^{sec.}\}_{j\in\mathcal{J}}=\emptyset$

Linking Gauge Theories and Symmetry Trading

$$\begin{tikzcd}
\text{partial gauge fixing } \phi_\alpha = 0 \arrow[d] & \boxed{\begin{array}{c}\text{Linking Theory}\\ \Gamma(q,p,\phi,\pi)\\ \chi_1^\alpha, \chi_\alpha^2, \chi_3^\mu\end{array}} \arrow[l] \arrow[r] & \text{partial gauge fixing } \pi^\alpha = 0 \arrow[d] \\
\boxed{\begin{array}{c}\text{Theory A}\\ \Gamma(q,p), \chi_\alpha^2, \chi_3^\mu\end{array}} \arrow[d] & & \boxed{\begin{array}{c}\text{Theory B}\\ \Gamma(q,p), \chi_1^\alpha, \chi_3^\mu\end{array}} \arrow[d] \\
\text{gauge fixing } \pi_o^\alpha(q,p) = 0 \arrow[r] & \boxed{\begin{array}{c}\text{Dictionary}\\ \Gamma_{red}, \chi_3^\mu\end{array}} & \text{gauge fixing } \phi_\alpha^o(q,p) = 0 \arrow[l]
\end{tikzcd}$$

where $\chi_1^\alpha = 0$ is equiv. $\pi^\alpha = \pi_o^\alpha(q,p)$ and $\chi_\alpha^2 = 0$ is equiv. $\phi_\alpha = \phi_\alpha^o(q,p)$

Equivalence of Gauge Theories on $(\Gamma, \{.,.\})$

$\exists$ gauge fixing with identical trajectories (same IVP and EOM)

Canonical Best Matching (an implementation of J. Barbour's Machian ideas)

Goal: Implement symmetry $q_i \to Q_i(q, \phi)$

first class system $(\Gamma, \{.,.\}, H, \{\chi^\mu\}_{\mu \in M})$, group param. ϕ_α

Construction

- phase space extension $\Gamma \to \Gamma \times \Gamma(\phi, \pi)$

equivalence with orig. Γ through first class constraints $\pi^\alpha \approx 0$

- canonical transformation (generator) $F = Q_i(q, \phi) P^i + \phi_\alpha \Pi^\alpha$

takes $\pi^\alpha \to \pi^\alpha - \pi_o^\alpha(q, p)$

- Impose best matching condition $\pi^\alpha = 0$:

(1) π^α commutes ⇒ orig. system had gauge invariance
(2) π^α gauge fixes some χ^μ ⇒ equivalence of gauge theories
(3) π^α generates secondary constraints ⇒ complete Dirac procedure

in general one obtains mixture of these three cases

Canonical General Relativity (you know that)

ADM formulation

- Global hyperbolicity; for now also Σ is compact without boundary
- Poisson bracket $\{F(g), \pi(f)\} = F(f)$
- spatial diffeomrophism constraints $H(v) = \int_\Sigma d^3x \pi^{ab} (\mathcal{L}_v g)_{ab}$
- local refoliation constraints

$$S(N) = \int_\Sigma d^3x N \left(\frac{1}{\sqrt{|g|}} \pi^{ab} G_{abcd} \pi^{cd} - \sqrt{|g|}(R - 2\Lambda) \right)$$

Generic IVP and Regularity:

are strictly proven in $\pi = g_{ab}\pi^{ab} = const.$ gauge. (Only few extensions.)

Linking GR and Shape Dynamics

ADM best matched w.r.t. VPCT

- extend phase space by canonical pair $\phi(x), \pi_\phi(x)$
- generating functional $F = \int d^3x \left(g_{ab} e^{4\hat{\phi}} \Pi^{ab} + \phi \Pi_\phi \right)$, where $\hat{\phi} = \phi - \frac{1}{6} \ln \langle e^{6\phi} \rangle_g$

canonical transformation $T : \pi_\phi \mapsto \pi_\phi - 4 \left(\pi - \langle \pi \rangle_g \sqrt{|g|} \right)$

- impose $\pi_\phi = 0$ and work out reduced phase space
- volume preserving condition leaves one Hamilton constraint behind

$\Rightarrow$ **matching trajectories** (SD in ADM-gauge = GR in CMC-gauge)

Shape Dynamics on $(\Gamma_{ADM}, \{.,.\}_{ADM})$

diffeomorphism: $H(\xi) = \int d^3x \pi^{ab} \mathcal{L}_\xi g_{ab}$

loc. conformal trf.: $D(\rho) = \int d^3x \rho(\pi - \langle \pi \rangle_g \sqrt{|g|})$

Hamiltonian: $H_{SD} = \int d^3x \; T S_{ADM}(x)|_{\phi=\phi_o(g,\pi)}$

where ϕ_o satisfies inhomogeneous Lichnerowicz-York equation and $\langle e^{6\phi} \rangle_g = 1$ ($\exists$ existence proof).

Pure Shape Dynamics (trajectories vs. IVP)

Dirac algebra is traded for **Shape Dynamics algebra**

$$\begin{array}{l|l}
[S(N_1),S(N_2)] = H(N_1\nabla N_2 - N_2\nabla N_1) & [D(\rho_1),D(\rho_2)] = 0 \\
 & [D(\rho),H_{SD}] = 0 \\
[H(\xi),S(N)] = S(\mathcal{L}_\xi N) & [H(\xi),D(\rho)] = D(\mathcal{L}_\xi \rho) \\
 & [H(\xi),H_{SD}] = 0 \\
[H(\xi_1),H(\xi_2)] = H([\xi_1,\xi_2]) & [H(\xi_1),H(\xi_2)] = H[\xi_1,\xi_2]
\end{array}$$

- Nonlin. constraints traded for linear constraints and nonloc. Hamiltonian
- Implies different Theory Space for Quantum Gravity.

for IVP (fixed time slice): lift volume preservation

- trade $S(N)$ for $D(\rho) = \int_\Sigma d^3x\rho\left(\pi \pm \sqrt{8\Lambda}\sqrt{|g|}\right)$.
- locality, but **frozen** dynamics (fixed time slice)

Objection 1: “You have no Spacetime Picture.”

Answer: No.

Use a minimally coupled test multiplet (as clocks/rods) to recover spacetime operationally.
Relationalist: You should do the same in GR.

Real Problem:

Universality of recovered spacetime (all fields see same spacetime).

Possible way out:

Symmetry Doubling: SD and GR BRST-invariances at the same time.
(this is work currently in progress, see “Outlook”)

Coupling Matter to Shape Dynamics

Construction

- Best matching of ADM+matter system w.r.t. VPCT
- Shape Dynamics through phase space reduction

⇒ **Equivalence** with GR+matter system by construction

Conformal weight for matter?

- Constraint-decoupling (conf., diffeo., gauge) as guiding principle

⇒ **Solution** for bosonic Standard Model: conformal weight 0 for all matter

$$F = \int_\Sigma d^3x \left(e^{4\hat{\phi}} g_{ab}\Pi^{ab} + \phi\Pi_\phi + A^i_a \mathcal{E}^a_i + \varphi^\alpha \Pi_\alpha + \dots \right)$$

Shape Dynamics-matter system:

- ADM $S(N)$ ⇒ SD-Ham. H_{SD} and $Q(\rho) = \int_\Sigma \rho \left(\pi - \langle\pi\rangle \sqrt{|g|} \right)$
- diffeomorphism- and gauge constraints are unaltered
- restrictions on $\langle\pi\rangle$ **only** from cosmological constant and Higgs potential

Objection 2: "It's just the York procedure."

Key differences with York procedure

- York procedure concerns only IVP (decoupling of constraints), SD concerns dynamics (propagation of constraints)
  $\Rightarrow$ Hamiltonian needs lapse fixing equation (York) vs. Lichnerowicz-York-equation (SD).
- Volume preserving condition (**not** used by York) is essential for equivalence of trajectories and gives correct scaling.
- Best matching is **canonical** transformation (SD) vs. manual TT-decomposition (York)

Summary:

The existence of SD is based in the same existence theorems as York, but the construction and consequences are fundamentally different.

Objection 3: "Your Hamiltonian is Unmanageable."

Yes, pertaining to classical trajectories. But:

• classically, one can work in ADM gauge then $H_{SD} = H_{ADM}(N \equiv 1)$
• once one is interested in **generic** IVP solution one needs to fix gauge also in ADM
then ADM suffers from the same complications

H_{SD} can be constructed explicitly in:

- $2+1$-dimensions (on sphere and torus)
- Strong gravity limit (precisely: whenever spatial derivatives negligible)
- perturbation theory around ADM solutions (in particular cosmological perturbation theory, WIP)

Gravity in 2+1 Dimensions

One of the 2+1 Tricks

$g = e^{2\lambda} f^* g(\tau)$ and TT-decomp. of π reduces GR to Teichmüller-sp dyn.

SD on 2+1 Torus

- lin. constr. $H(v) = \int_T d^2x \pi^{ab} (\mathcal{L}g)_{ab}$, $D(\rho) = \int_T d^2x \rho(\pi - \langle\pi\rangle \sqrt{|g|})$
- SD-Hamilton constraint $H_{SD} = \frac{\tau^2}{2V}\left(p_1^2 + p_2^2\right) - \frac{V}{2}\left(\langle\pi\rangle^2 - 4\Lambda\right)$

Dirac Qunat. = Reduced phase space Quant.

- Set $\hat{\pi}^{ab} = i\hbar \frac{\delta}{\delta_{g_{ab}}} \Rightarrow$ linear constraints imply $\psi[g[\lambda, f; \tau_1, \tau_2)] = \psi(\tau_1, \tau_2)$

$\Rightarrow H_{SD} = -\tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2) + V^2(\partial_V^2 + 4\Lambda)$ (equal to reduced phase space Hamiltonian)

Large CMC-Volume Expansion

- genus $\geq 2 \Rightarrow V/V_o$-expans.: $H_{SD} = -\frac{V}{2}(\langle\pi\rangle^2 - 4\Lambda) - R + \mathcal{O}(\frac{1}{V})$
- works in higher dimensions (large CMC-volume/CFT-correspondence)

Objection 4: “Non-CMC solutions to Einstein’s equations.”

Answer:

This is a restriction compared to GR (like global hyperbolicity).

However:

Boundaries are in principle treatable (see future projects).
If dynamical boundaries are also treatable then one could possibly circumvent some of these restrictions.

Shape Dynamics in Ashtekar-Barbero variables

Road map

triad var. $(e^i_a, \pi^a_i) \Rightarrow$ ext. curv. var. $(K^i_a, E^a_i) \Rightarrow$ Ashtakar var. (A^i_a, E^a_i)

Triad vars.: (e^i_a, π^a_i), rotation const. $G(\lambda) = \int_\Sigma d^3x \lambda^i \epsilon_{ij}{}^k e^k_a \pi^a_k$

- best matching generator $F = \int_\Sigma d^3x \left(e^i_a e^{2\hat{\phi}} \Pi^a_i + \phi \Pi_\phi \right)$
- linear constr.: $H(v)$, $C(\rho)$ and $G(\lambda)$
- Hamiltonian: $H_{SD} = TS(N \equiv 1)|_{\phi = \hat{\phi}_o}$

$\Rightarrow (K^i_a, E^a_i)$ and can. trf. $F = \int d^3x (K^i_a \beta \tilde{E}^a_i + \tilde{E}^a_i \Gamma^i_a(\tilde{E}))$

Ashtekar vars.: (A^i_a, E^a_i), Gauss constraint $G(\lambda)$

- linear constraints: $H(v)$ and $G(\lambda)$
- VPCT-generator: $C(\rho) = \int_\Sigma d^3x \rho ((A^i_a - \Gamma^i_a)\tilde{E}^a_i - \langle (A^i_a - \Gamma^i_a)\tilde{E}^a_i \rangle \sqrt{|\tilde{E}|}\ )$
- Hamiltonian: $H_{SD} = TS(N \equiv 1)|_{\phi = \hat{\phi}_o}$ (retain Gauss-constr.)

Kinematic Loop Quantization

Do everything as in kinematic LQG:

- dense orthogonal set $T(A) = \prod_e \rho^{j_e}_{n_e m_e}(h_e(A))$
- holonomy-matrix elements act by multiplication $\rho^{j}_{nm}(h(A))$
- fluxes act as derivations $E(S, \lambda)$
- solve Gauss- and diffeomorphism constraint

$\Rightarrow$ orthogonal set: gauge-inv. spin-knot functions

classical VPCT-invariants

- VPCTs of holonomies are unmanageable
- VPCTs of fluxes are straightforward: $E^i_a(x) \mapsto e^{4\hat{\phi}(x)} E^i_a(x)$

$\Rightarrow$ total volume V and all angles (=gauge-inv. ratios) are **invariant**

$\Rightarrow$ all nonvanishing areas are **pure gauge**

Disclaimer

Warning

The following is a **naive** application of LQG methods to the IVP of SD only.

In particular be **critical** of:

1. No quantization of equivalent dynamics
2. Heuristic treatment of conf. trfs.

Tentative Quantization

Heuristic implementation of VPCTs

- choose recoupling basis ⇒ max. comm. set of angles
- determine SNF uniquely by (area-,angle-) eignevalues and keep V_{tot}

⇒ areas are pure gauge ⇒ choose unique representative

⇒ knot, max. set of angles and V_{tot} label basis

- How quantize H_{SD} ???

For total Dirac procedure (fix time slice)

- trade **also** H_{SD} for $\langle\pi\rangle - \alpha V$ (can. trf. to $\langle\pi\rangle$)

⇒ basis labeled by knot, max. set of angles

Problem:

Can one get rid of recoupling choice?

A Questionable Interpretation

Objection:

Where is equivalence with CMC dynamics?

Large CMC-volume correpsondence

- assume $\Lambda > 0$ and suitable initial data $\Rightarrow$ asymptotic expansion
- use V/V_o-expansion $\Rightarrow H_{SD} \rightarrow 12\Lambda - \langle\pi\rangle^2$

$\Rightarrow$ classical equivalent $\langle\pi\rangle \approx \pm\sqrt{12\Lambda}$

Questionable Interpretation

"The states in the mutual constraint kernel describe asymptotic shapes of the universe in the large volume limit."

Summary

Features of Shape Dynamics

1. Shape Dynamics arises from trading refoliation invariance for local conformal invariance
2. Linking theories and canonical best matching provide general mechanism for symmetry trading
3. Existence theorems of for IVP ensure Shape Dynamics exists
4. SD on 2+1-torus in metric variables can be quantized without phase space reduction
5. large CMC-volume/CFT-correspondence
6. Works with standard matter content
7. Spacetime picture can be recovered operationally
8. tentative LQG-quantization

Outlook

Current Project: **Symmetry Doubling**

Observation: special $H_{BRST} = \sigma^\alpha \chi_\alpha + \ldots$ is BRST invariant under **both** $\Omega_{orig} = \eta^\alpha \chi_\alpha + \ldots$ and $\Omega_{dual} = \bar{\eta}_\alpha \sigma^\alpha + \ldots$
$\Rightarrow$ small theory space
here: complete symmetry trading is admissible $\Rightarrow$ locality

There are many open questions (examples):

- Symplectic geometry of symmetry trading/symmetry doubling
- Treatment of boundaries (e.g. isolated horizons)
- More advanced: dynamical boundaries (possibly non CMC?)
- Higher orders in perturbation theory
- SD-gauge transformations of solutions to GR
- Symmetry trading/symmetry doubling in other theories

Thank you!