Volume operator in Loop Quantum Cosmology

Wojciech Kamiński

Uniwersytet Warszawski

19.11.2019

[arxiv: 1906.07554],

Introduction

FRW model with massless scalar field

- Symmetry reduction of gravity coupled to the massless scalar field

$$\{\phi, p_\phi\} = 1, \quad \{c, v\} = 1, \tag{1}$$

- Only one constraint left

$$\frac{p_\phi^2}{v} - vc^2 = 0 \tag{2}$$

- Solving constraint (deparametrization through the scalar field)

$$p_\phi = \pm\sqrt{v^2c^2} = \pm|vc|, \quad \Theta = v^2c^2 \tag{3}$$

- Superselection + (we will see the problem ...)

LQC models

What is Loop Quantum Cosmology:

- LQC is not symmetry reduction of Loop Quantum Gravity, but inspired quantization of homogeneous mini super space.
- Can we obtain it from LQG?
  [Engle, Fleischhack, Hanusch, Thiemann, Vilensky...]
- As relation to LQG unclear, can we trust that predictions of LQC still holds in LQG?

Importance of LQC

- It serves as a testing ground for LQG
- It provides effective geometries for cosmological computations (CMB , dressed metric)
  [Agullo, Ashtekar, Dapor, Lewandowski, Singh, ...]

LQC models

The Hilbert space (after symmetry reduction $v \to -v$)

$$\mathcal{H} = \left\{ f\colon \mathbb{Z}_+ \to \mathbb{C}, \sum_v B(v)|f(v)|^2 < \infty \right\}$$

Operator $\hat{H}_{LQC} = +\sqrt{\hat{\Theta}_{LQC}}$

$$\hat{\Theta}_{LQC} = -B(v)^{-1}\left(C(v)\hat{h}_{+1} + C_0(v) + C(v-1)\hat{h}_{-1}\right)$$

where $\hat{h}_{\pm 1}$ are shifts by 1.

Case $\Lambda = 0, \quad k = 0$

In what follows only asymptotic expansions of C, C_0 and B matter.

New model [Yang, Ding, Ma], [Dapor, Liegener] $\hat{\Theta}_{LQC}$ 5-term difference equation.

LQC models

The Hilbert space (after symmetry reduction $v \to -v$)

$$\mathcal{H} = \left\{ f \colon \mathbb{Z}_+ \to \mathbb{C}, \sum_v B(v)|f(v)|^2 < \infty \right\}$$

Operator $\hat{H}_{LQC} = +\sqrt{\hat{\Theta}_{LQC}}$

$$\hat{\Theta}_{LQC} = -B(v)^{-1}(C(v)\hat{h}_{+1} + C_0(v) + C(v-1)\hat{h}_{-1})$$

where $\hat{h}_{\pm 1}$ are shifts by 1.

Case $\Lambda = 0, \quad k = 0$

In what follows only asymptotic expansions of C, C_0 and B matter.

New model [Yang, Ding, Ma], [Dapor, Liegener] $\hat{\Theta}_{LQC}$ 5-term difference equation.

Details skipping constants...

$$\hat{\Theta}_{LQC} = -B(v)^{-1}(C(v)\hat{h}_{+1} + C_0(v) + C(v-1)\hat{h}_{-1})$$

with coefficients admitting expansion in v^{-1} with first terms

$$C(v) = v + \frac{1}{2} + a + \frac{b}{v} + O(v^{-2}) \quad (4)$$

$$C_0(v) = -2v - 2a - \frac{2b}{v} + O(v^{-2}) \quad (5)$$

$$B(v) = \frac{1}{v} + O(v^{-2}) \quad (6)$$

Covers [Ashtekar, Pawłowski, Singh],
[Mena-Marugan, Martin-Benito, Olmedo]
[Ashtekar, Corichi, Singh],...

Semi-classical dynamics

Observations about the limits

- States peaked on high energy are also peaked on high volumes. In the Fourier transform picture

$$L^2(S^1), \quad \hat{v} = i\partial_c, \quad \hat{h}_{+1} = e^{ic} \tag{7}$$

  it corresponds to high momenta (similar to large j limit in spin foams).
- Moreover large v limit also appears at late time.
- Limit of physical interests $c \to 0$ (small curvature).

Semi-classical dynamics

Semi-classical dynamics [Bojowald], [Taveras]

- Define Θ_{eff} as an expectation value in suitable coherent state peaked at (v, c) (**ambiguity**)

$$\Theta_{eff} = 4v^2 \sin^2 \frac{c}{2} + O(1)? \tag{7}$$

- It can be computed by naive replacement

$$\hat{v} \to v, \quad \hat{h}_{+1} \to e^{ic} \tag{8}$$

the ordering ambiguity gives $O(v)$.

Semi-classical dynamics

Elliptic case:

- Semi-classical behaviour captured in effective dynamics for $\Lambda < 0$
$$\Theta_{eff}^{\Lambda} = v^2 \left(4 \sin^2 \frac{c}{2} - \Lambda\right) + O(v) \qquad (7)$$
when coefficient at v^2 always nonzero (PDO).
- The details of the classical evolution for $\Lambda = 0$, $k = 0$ depend on $O(v)$ for large volume (late times), but not close to the bounce (elliptic region).

Semi-classical dynamics

- Can we always trust semi-classical dynamics in elliptic region? Is it in semi-classical limit **local** like the classical dynamics?
- Can we extend it to the late time asymptotics?
- Numerical studies: States peaked on high energies follow semi-classical trajectories $(c(t), v(t))$

$$\pm\sqrt{\Theta_{eff}} = \pm 2|v|\left|\sin\frac{c}{2}\right| \tag{7}$$

Semi-classical dynamics

- Can we always trust semi-classical dynamics in elliptic region? Is it in semi-classical limit **local** like the classical dynamics?
- Can we extend it to the late time asymptotics?
- Numerical studies: States peaked on high energies follow semi-classical trajectories $(c(t), v(t))$

$$\pm\sqrt{\Theta_{eff}} = \pm 2|v|\left|\sin\frac{c}{2}\right| \tag{7}$$

- **Is it really true?** [Dapor, WK, Liegener, in progress]

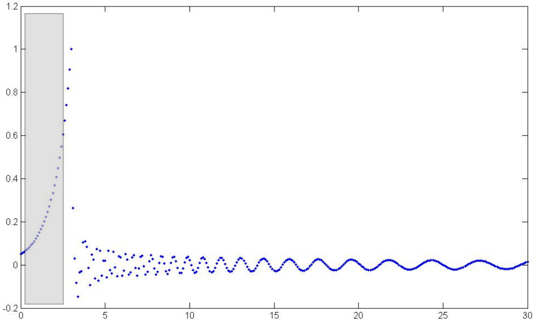
Semi-classical hamiltonian

$$\pm\sqrt{\Theta_{eff}} = \pm 2|v|\left|\sin\frac{c}{2}\right| \tag{7}$$

- We can attack evolution problem directly [Bojowald], [Bojowald, Skirzewski], [Ashtekar, Corich, Singh], [Dapor, WK, Liegener in progress].
  Better?
- Consider asymptotic behaviour of the eigenfunctions of $\hat{\Theta}$ and derive properties of evolution afterwards [Ashtekar, Pawlowski, Singh], [WK, Pawlowski].
  Better developed for LQC.

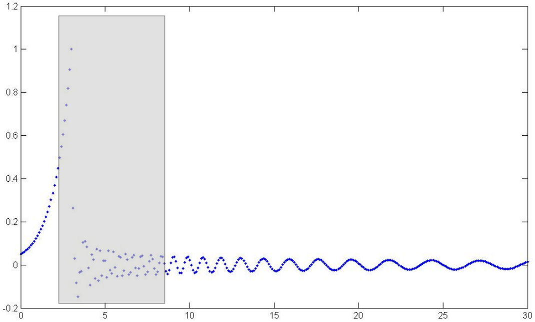
Main result

Eigenfunctions $\hat{\Theta} e_\omega = \omega^2 e_\omega$ (reminder)



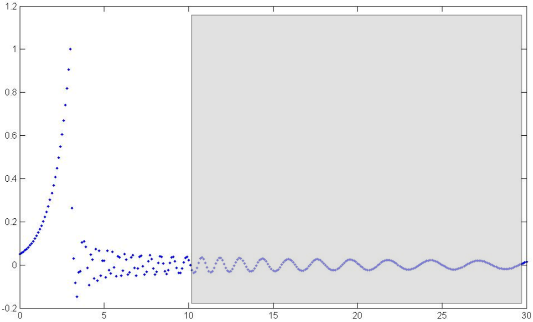
- $v^{|\omega|}$ **expansion for large energies** ω
- Turning point (large energies) better description in Fourier representation
- Asymptotic behaviour for large volume v, all $\omega \neq 0$. Sensitive to the details of the hamiltonian.

Eigenfunctions $\hat{\Theta} e_\omega = \omega^2 e_\omega$ (reminder)



- $v^{|\omega|}$ expansion for large energies ω
- **Turning point (large energies) better description in Fourier representation**
- Asymptotic behaviour for large volume v, all $\omega \neq 0$. Sensitive to the details of the hamiltonian.

Eigenfunctions $\hat{\Theta} e_\omega = \omega^2 e_\omega$ (reminder)



- $v^{|\omega|}$ expansion for large energies ω
- Turning point (large energies) better description in Fourier representation
- **Asymptotic behaviour for large volume v, all $\omega \neq 0$. Sensitive to the details of the hamiltonian.**

Transfer matrix [Ashtekar, Pawlowski, Singh], [WK, Pawlowski]

Compute $b_n(\omega)$ for $\omega \in \mathbb{C} \setminus i\mathbb{Z}$ order by order

$$d_\omega(v) = exp\left(\sum_{n=1} \frac{b_n(\omega)}{v^n}\right) . \qquad (8)$$

such that (as a series in v^{-1})

$$C(v)d_\omega(v+1) + (B(v)\omega^2 + C_0(v)) + C(v-1)d_\omega^{-1}(v) = 0 \qquad (9)$$

There are two solutions $d_\omega^{\pm}(v)$.

$$b_1^{\pm}(\omega) = \pm i\omega \qquad (10)$$

Compute $b_n(\omega)$ for $\omega \in \mathbb{C} \setminus i\mathbb{Z}$ order by order

$$d_\omega(v) = exp\left(\sum_{n=1} \frac{b_n(\omega)}{v^n}\right). \qquad (8)$$

Taking finite truncation we define

$$\phi_N^{\pm} = \prod_{v'=1}^{v} d_{\omega,N}^{\pm}(v'), \quad (\hat{\Theta} - \omega^2)\phi_N^{\pm} = O(v^{-N}) \qquad (9)$$

Transfer matrix [Ashtekar, Pawlowski, Singh], [WK, Pawlowski]

Compute $b_n(\omega)$ for $\omega \in \mathbb{C} \setminus i\mathbb{Z}$ order by order

$$d_\omega(v) = exp\left(\sum_{n=1} \frac{b_n(\omega)}{v^n}\right). \quad (8)$$

Transfer matrix M

- (d =number of terms$-$1) approximate solutions, M
- error $\times \|M^{-1}\|$ is summable

then **there exist solutions with given asymptotics.**

We have two solutions (without conditions at 0) with asymptotics

$$v^{\pm i\omega}(1 + O(v^{-1})) \quad (9)$$

Properties of the solutions

The solution to

$$\hat{\Theta} e_\omega(v) = \omega^2 e_\omega(v), \quad e_\omega(1) = 1 \tag{10}$$

satisfying symmetric condition at 0

- in our case asymptotics of solutions $v^{\pm i\omega}$.
- for any $\omega \notin i\mathbb{Z}$ satisfies

$$|e_\omega(v)| = O\left(v^{|\Im\omega|}\right) \tag{11}$$

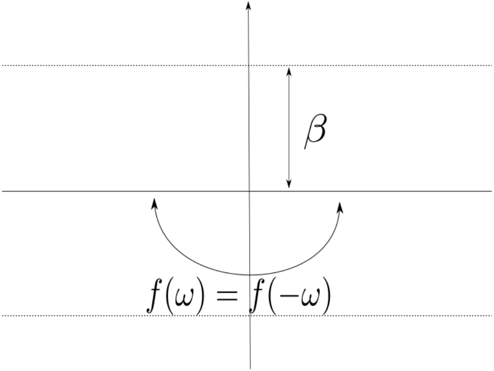
- Moreover, for ω real they are **generalized eigenfunctions** for positive part of the spectrum of $\hat{\Theta}$ (spectrum $\mathbb{R}_+$)

Argument

Let us assume that $\psi \in D(\hat{v}^\beta)$ for $\beta > 0$. The function

$$f(\omega) = \langle \psi, e_\omega \rangle := \langle v^\beta \psi, v^{-\beta} e_\omega \rangle, \quad f(\omega) = f(-\omega) \qquad (12)$$

is holomorphic in a strip $\{z \in \mathbb{C} \colon |\Im z| < \beta\} \setminus i\mathbb{Z}$

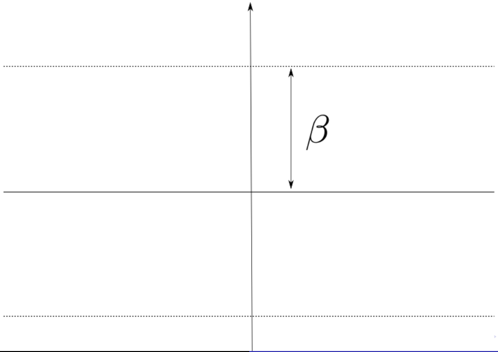


Argument

If $e^{it\sqrt{\hat{\Theta}}}\psi \in D(\hat{v}^{\beta})$ then the same is true for the function

$$\tilde{f}(\omega) = \langle e^{it\sqrt{\hat{\Theta}}}\psi, e_{\omega}\rangle \tag{12}$$

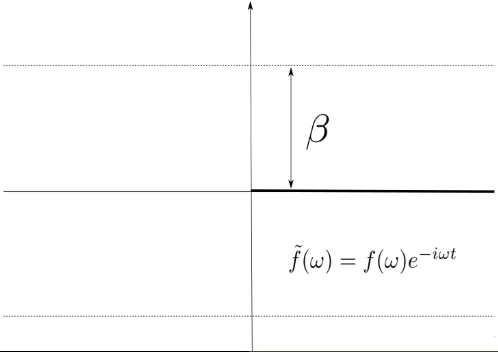
- From eigenfunction expansion $\tilde{f}(\omega) = e^{-it\omega} f(\omega)$ for $\omega \in \mathbb{R}_{+}$
- The analytic extension $e^{-it\omega} f(\omega)$ is not symmetric.
- ...unless $f(\omega) = 0$ and $\hat{\Theta}\Psi = 0$



If $e^{it\sqrt{\hat{\Theta}}}\psi \in D(\hat{v}^{\beta})$ then the same is true for the function

$$\tilde{f}(\omega) = \langle e^{it\sqrt{\hat{\Theta}}}\psi, e_\omega \rangle \tag{12}$$

- **From eigenfunction expansion** $\tilde{f}(\omega) = e^{-it\omega} f(\omega)$ **for** $\omega \in \mathbb{R}_+$
- The analytic extension $e^{-it\omega} f(\omega)$ is not symmetric.
- ...unless $f(\omega) = 0$ and $\hat{\Theta}\Psi = 0$

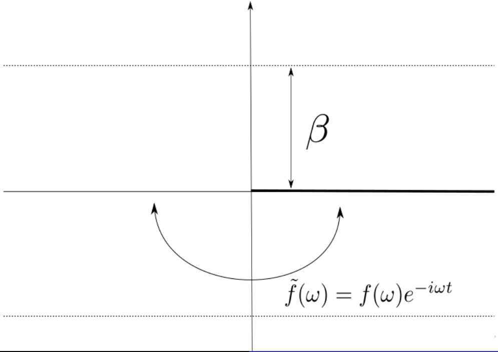


Argument

If $e^{it\sqrt{\hat{\Theta}}}\psi \in D(\hat{v}^{\beta})$ then the same is true for the function

$$\tilde{f}(\omega) = \langle e^{it\sqrt{\hat{\Theta}}}\psi, e_\omega \rangle \tag{12}$$

- From eigenfunction expansion $\tilde{f}(\omega) = e^{-it\omega} f(\omega)$ for $\omega \in \mathbb{R}_+$
- **The analytic extension $e^{-it\omega} f(\omega)$ is not symmetric.**
- ...unless $f(\omega) = 0$ and $\hat{\Theta}\Psi = 0$

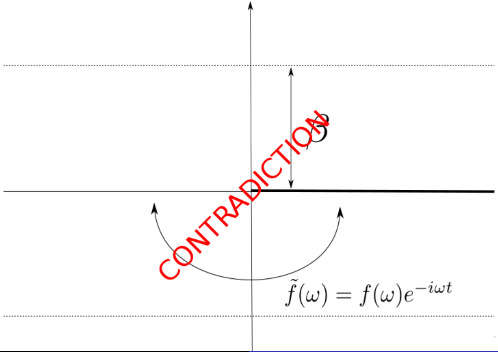


Argument

If $e^{it\sqrt{\hat{\Theta}}}\psi \in D(\hat{v}^{\beta})$ then the same is true for the function

$$\tilde{f}(\omega) = \langle e^{it\sqrt{\hat{\Theta}}}\psi, e_\omega \rangle \tag{12}$$

- From eigenfunction expansion $\tilde{f}(\omega) = e^{-it\omega} f(\omega)$ for $\omega \in \mathbb{R}_+$
- The analytic extension $e^{-it\omega} f(\omega)$ is not symmetric.
- **...unless** $f(\omega) = 0$ **and** $\hat{\Theta}\Psi = 0$



A puzzling result

The result

State $\Psi \in D(\hat{v}^\beta)$ can stay in $D(\hat{v}^\beta)$ under the evolution only if it is supported on nonpositive spectrum.

In APS and MMO it means that $\Psi = 0$.

- The problem with the volume was suspected before [Varadarajan' 08], etc

Questions:

- Why it was not noticed in numerical simulations?
- Tension with results from the exactly solvable models like [Ashtekar, Corichi, Singh]

Example from WdW

Similar result can be proven in the context of WdW model

$$\Theta = -(v\partial_v)^2, \quad \mathcal{H} = L^2(R_+, v^{-1}dv) \tag{13}$$

Change of variables $x = \ln v \Rightarrow$ Klein-Gordon equation

positive momenta right moving, negative momenta left moving.

Let us consider a Gaussian state

$$\hat{\Psi}_t(p) = e^{i|p|t}\hat{\Psi}(p) \tag{14}$$

Evolved state is **non-smooth** at $p = 0$.

Fourier transform

Fourier transform dominated by $p = 0$ part

$$\Psi_t(x) = \frac{C_t}{x^2} + O(x^{-3}), \quad C_t \approx \hat{\Psi}'_t(0) \tag{15}$$

- $\Psi_t(v) = \frac{C_t}{\ln^2 v} + O(\ln^{-3} v) \notin D(\hat{v}^{\beta})$
- However $\hat{\Psi}'_t(0) \approx e^{-\sigma p_0^2}$ very small for cases used in numerical studies.

Conjecture

Non-integrable part is so small that it is invisible in the numerical simulations.

Solvable models

We can divide our state into a smooth part

$$\hat{\Psi}^s_t(p) = e^{ipt}\hat{\Psi}(p) \tag{16}$$

and the remainder

$$\hat{R}_t(p) = \begin{cases} 2i \sin pt \hat{\Psi}(p), & p < 0 \\ 0, & p \geq 0 \end{cases} \tag{17}$$

The remainder is small but responsible for the problem with the volume.

Solvable models

Assumption that R_t can be omitted.

Now we know that it is not allowed

- ... but the evolved state still nicely peaked at the semi-classically evolved $v_0(t)$, however not in the sense of expectation values.
- The expectation value of $\hat{v}$ in Ψ_t^s follows semi-classical trajectory (proposition for definition of the semi-classical volume?).

Locality

The evolution seems to depend on the hamiltonian far from the region in which we evolve

$$2v\sin\frac{c}{2}, \quad 2|v|\left|\sin\frac{c}{2}\right|, \tag{16}$$

Semi-classical dynamics $\longleftrightarrow$ Quantum dynamics
local non-local in phase space.

Maybe still semi-local in some class of bounded observables?

- In fact already $\langle \ln v_t \rangle$ well defined.

Conclusions

Physical consequences

Dressed metric approach [Ashtekar, WK, Lewandowski], [Agullo], [Dapor, Lewandowski] etc.

- Quantum field evolves as in the effective metric.
- Based on approximations (hard to justify)
- This metric is expressed through $\langle \hat{v}^{\beta} \rangle_t = \infty$.

Questions

1. Can we trust these approximations?
2. What we should place instead of ill-defined quantities?

Work in progress [Kolanowski, WK, Lewandowski, in prep.]

Outlook

- The evolution of the volume is ill-defined,
- It is an issue for $k = 0$ and $\Lambda = 0$ model
  1. not clear what happens for $\Lambda > 0$,
  2. the problem disappears for $\Lambda < 0$,
  3. It is probably also present in recent models DL-YDM
- The reason is restriction to $+\sqrt{\Theta}$ sector.

Conjecture

Some versions of group averaging may lead to non-problematic evolution (but observable will mix sectors)

Outlook

- The evolution of the volume is ill-defined,
- It is an issue for $k = 0$ and $\Lambda = 0$ model
  1. not clear what happens for $\Lambda > 0$,
  2. the problem disappears for $\Lambda < 0$,
  3. It is probably also present in recent models DL-YDM
- The reason is restriction to $+\sqrt{\Theta}$ sector.

Conjecture

Some versions of group averaging may lead to non-problematic evolution (but observable will mix sectors)

Thank you!