Asymptotic Dynamics of Holonomy Spin Foam Models

Frank Hellmann
with Benjamin Bahr, Bianca Dittrich, Wojciech Kaminski

Albert Einstein Institute, Golm

September 4th, 2012

Papers:

- B. Bahr, B. Dittrich, FH, W. Kaminski:

  Holonomy Spin Foam Models: Definition and coarse graining. (arxiv:1208:3388),

  Holonomy Spin Foam Models: Boundary Hilbert spaces and canonical dynamics. (arxiv:soon)

- FH, W. Kaminski:

  Holonomy Spin Foam Models: Asymptotic Dynamics of EPRL type models. (arxiv: soon+ϵ)

Alesci, Baez, Barrett, Baratain, Bonzom, Conrady, Crane, Ding, Engle, Fairbairn, Freidel, Han, Kisielowski, Krasnov, Lewandowski, Livine, Oeckel, Oriti, Pereira, Perez, Pfeiffer, Reisenberger, Rovelli, Smerlak, Speziale, Wieland, Zhang, ...[1] [since 1998, ongoing].

[1] Apologies to everyone I forgot.

Holonomies throughout History

Holonomy formulations have been central to understanding spin foams throughout their development.

90s Ooguri, Boulatov, Reisenberger, Rovelli

00s Perez, Oriti, Williams, Pfeiffer, Oeckl

10s Rivasseau, Gurau, Ben Geloun, Bonzom, Smerlak

...

The main results

The Holonomy Spin Foam Partition Function

$$\mathcal{Z}_\star(\mathcal{C}) = \int \left(\prod_{e\subset f} \mathrm{d}g_{ef}\right)\left(\prod_{v\subset e} \mathrm{d}g_{ev}\right)\left(\prod_{e\subset f} E_\star(g_{ef})\right)\left(\prod_{f} \delta(g_f)\right).$$

with $\star \in \{\mathrm{BC}, \mathrm{BF}, \mathrm{BO}^\gamma, \mathrm{EPRL}^\gamma, \mathrm{FK}^\gamma, \dots\}$, not KKL.

- Maximal number of structural assumptions = Minimal number of parameters.
- Natural common boundary space for all models.
- Immediate generalisation to finite groups (without studying complicated representation theory).
- Composition of distribution is very geometric, ill defined. Use tools from distribution theory to study partition function.

The main results - 2

The Holonomy Spin Foam Partition Function

$$\mathcal{Z}_\star(\mathcal{C}) = \int \left(\prod_{e\subset f} \mathrm{d}g_{ef} \right) \left(\prod_{v\subset e} \mathrm{d}g_{ev} \right) \left(\prod_{e\subset f} E_\star(g_{ef}) \right) \left(\prod_{f} \delta(g_f) \right).$$

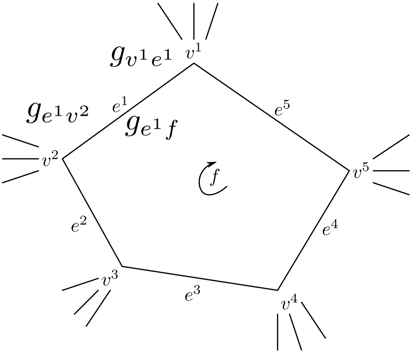
with $\star \in \{\mathrm{BC}, \mathrm{BF}, \mathrm{BO}^\gamma, \mathrm{EPRL}^\gamma, \mathrm{FK}^\gamma, \dots\}$, not KKL.

- Not well defined ($\prod \delta$) due to problematic configurations with arbitrarily large bivectors inside finite boundary.
- Any regularisation of the γ twisted models that preserves the geometricity of the connection is flat[2]. FK^0 is fine.
- This flatness has its origin in the incorrect twisting of the equations on the face. We can fix this by changing the type of face amplitude.

[2] Except for some accidental zeros.

The partition function - Space of parameters

$$\mathcal{Z}_\star(\mathcal{C}) = \int \left(\prod_{e\subset f} \mathrm{d}g_{ef}\right) \left(\prod_{v\subset e} \mathrm{d}g_{ev}\right) \left(\prod_{e\subset f} E_\star(g_{ef})\right) \left(\prod_{f} \delta(g_f)\right).$$



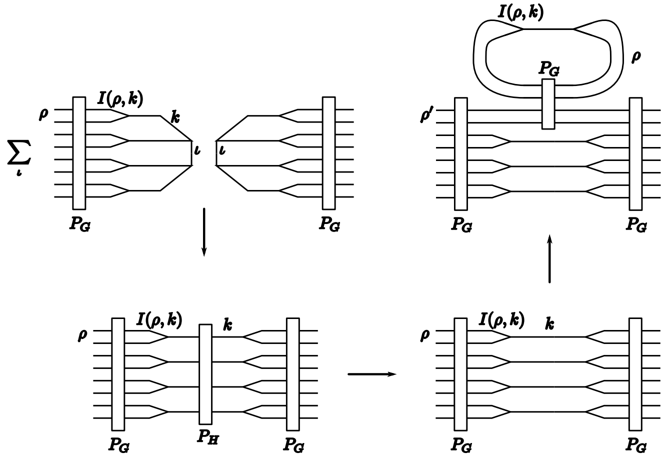
$$g_f = g_{v^1e^1}g_{e^1f}g_{e^1v^2}g_{v^2e^2}\dots g_{e^5v^1}$$
$$\tilde{g}_f = g_{v^1e^1}g_{e^1v^2}g_{v^2e^2}\dots g_{e^5v^1}$$

$H \subset G$, Lie or finite. $g \in G,\ h \in H$,

$$E(g) = E(g^{-1}),$$
$$E(hgh^{-1}) = E(g),$$

From now on: $G = \mathrm{Spin}(4)$, $H = \mathrm{SU}(2)$

The partition function - Equivalence to EPRL



Here ρ is a G Irrep, k is an H Irrep, ι is an H intertwiner and $I(\rho, k)$ is an injection.

The EPRL distribution

The EPRL simplicity distribution is thus

$$E^{\gamma}_{\mathrm{EPRL}}(g) = \sum_k \dim(\rho_k)\,\mathrm{tr}_{\rho_k}\left(D_{\rho_k}(g)\,I(\rho_k, k)I(\rho_k, k)^{\dagger}\right),$$

with $\rho_k = \left(\frac{1+\gamma}{2}k, \frac{|1-\gamma|}{2}k\right)$.

$\rightarrow$ Suggests generalization to finite groups[3]: Characterize the critical manifold M of E group theoretically. Mimic with Θ_M in the finite case.

[3] As the usual construction is all about the Lie algebra, this is hard otherwise!

The critical manifolds

Characterize the critical manifold M of E group theoretically:

- BF: $\mathbf{1}$
- BC: $\mathrm{SU}(2)_{diag} = H$
- EPRL$^{\gamma}$/FK$^{\gamma}$:
  $M_{\gamma} = \{g \in Spin(4) | \exists L \in \mathfrak{su}(2) : g = (\exp(L), \exp(-\gamma L))\}$.

Thus the generalizations to finite groups $H \subset G$ are:

- BF: $E(h) = \Theta_e(h)$ with e being the trivial subgroup consisting of the identity element.
- BC: $E(h) = \Theta_H(h)$
- EPRL^C/FK^C: $E_C(h) = \Theta_{M_C}(h)$, where C is a cyclic subgroup of G and $M_C = H \triangleright_{ad} C$.

Study renormalization numerically. First results: Non-trivial fix points on hierarchical lattices. Models like to flow to BF.

Analysis of distributions

Can be much more sophisticated in the study of distributions:

The wave front set of a distribution f (roughly)

The wavefront set $WF(f) \subset T^*G$ are the phase space elements (g, p) such that for semiclassical states $\psi(g, \lambda p)$ peaked on them, $\langle \psi(g, \lambda p) | f \rangle$ is not exponentially suppressed for $\lambda \to \infty$.

The wave front set of a distribution f (precisely)

Let f be a distribution over M, $f \in \mathcal{D}(M)$. The wave front set $WF(f) \subset T^*M$ of A is defined as the complement of the set of elements $\{(x, p) \in T^*M \setminus \{0\}\}$ such that there is a local coordinate patch $U \times V$ with

$$\exists U \times V \ni (x, p), \ \forall \phi \in C_0^\infty(U), \ \forall \tilde{p} \in V :$$

$$\int_U e^{i\lambda \tilde{p}\tilde{x}} \phi(\tilde{x}) f(\tilde{x}) \, d\tilde{x} = O(\lambda^{-\infty})$$

WF - Properties

These behave geometrically under composition:

- $f(g, g') = \tilde{f}(gg') \to WF(f)$ enforces good parallel transport between g and g', $p = g \triangleright p'$.
- $f(g) = \int \mathrm{d}g' f^1(g, g') f^2(g, g') f^3(g, g') \dots \to WF(f)$ is $WF(f^1) + WF(f^2) + WF(f^3) \dots$ plus closure:

$$p'_1 + p'_2 + p'_3 + \dots = 0.$$

Only well defined if

$$p_1 = p_2 = p_3 = \dots = 0 \implies p'_1 = p'_2 = p'_3 = \dots = 0$$

It can be shown that $WF(E_{\mathrm{EPRL}^\gamma})$ are the pairs (g, p) such that p is twisted simple and g is generated by a multiple of $*p$.

The wave front set of $\mathcal{Z}$

Main statement
$\langle\psi_\lambda|\mathcal{Z}\rangle$ is exponentially small unless ψ_λ is peaked on phase space points that occur as the boundary of the set of equations below.

At the vertex we have

$$\begin{aligned} p^v_{ee'} &= -p^v_{e'e}, \\ p^v_{ee'} &= g_{ve} \triangleright p^e_{vf}. \end{aligned}$$

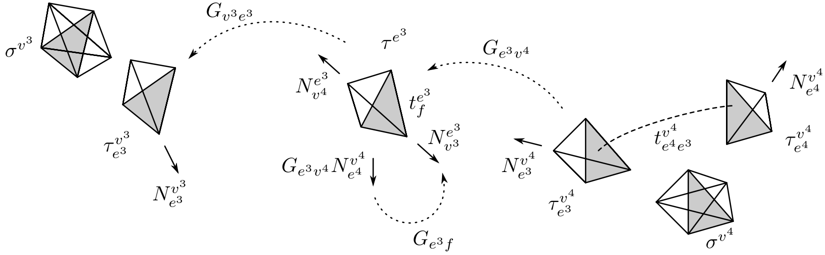
At the edge we obtain

$$\begin{aligned} p^e_{vf}&\text{is twisted simple} \\ \sum_{f\ni e} p^e_{vf} &= 0, \\ \exists \xi_{ef} \text{ s.t. } g_{ef} &= \exp(\xi_{ef} * \hat{p}^e_{vf}). \end{aligned}$$

And on the face we have

$$\begin{aligned} g_f \triangleright p^v_{e'e} &= p^v_{e'e}, \\ \tilde{g}_f \triangleright p^v_{e'e} &= p^v_{e'e}, \\ g_f = \mathbf{1} \quad &\text{or} \quad p^e_{vf} = 0. \end{aligned}$$

The geometric interpretation - 1



$$f = (\dots v^3, e^3, v^4, e^4 \dots)$$

$$\begin{aligned} t^e_f &= G_{ef}\, t^e_f \\ N^e_v = -N^e_{v'} &= G_{ef}\, G_{ev'} N^{v'}_{e'}. \end{aligned} \qquad \begin{aligned} t^e_f &= G_{ev'}\, t^{v'}_{ee'} \\ N^e_{v'} &= G_{ev'}\, N^{v'}_e. \end{aligned}$$

$$\text{with } (v, e, v', e') \subset f$$

The geometric interpretation - 2

Now with B^e_{vf} and $B^v_{ee'}$ the bivectors associated to t^e_{vf} and $t^v_{ee'}$ we have the following geometric equations:

At the vertex we have

$$\begin{aligned} B^v_{ee'} &= -B^v_{e'e}, \\ B^v_{ee'} &= G_{ve} \triangleright B^e_{vf}. \end{aligned}$$

At the edge we obtain

$$B^e_{vf} \text{ is simple}$$

$$\begin{aligned} \sum_{f \ni e} B^e_{vf} &= 0, \\ \exists \xi_{ef} \text{ s.t. } G_{ef} &= \exp(\xi_{ef} * \hat{B}^e_{vf}). \end{aligned}$$

And on the face we have

$$\begin{aligned} G_f \triangleright B^v_{e'e} &= B^v_{e'e}, \\ \tilde{G}_f \triangleright B^v_{e'e} &= B^v_{e'e}. \\ G_f &= \mathbf{1}. \end{aligned}$$

The geometric interpretation - Let's twist

Use $T_\gamma = \frac{1}{2}(1+\gamma *)$, invertible. Commutes with adjoint group action.
Acting on the equations above yields:

At the vertex we have

$$\begin{aligned} B^v_{ee'} &= -B^v_{e'e}, \\ B^v_{ee'} &= G_{ve} \triangleright B^e_{vf}. \end{aligned}$$

At the edge we obtain

$B^{\gamma e}_{vf}$ is twisted simple

$$\begin{aligned} \sum_{f \ni e} B^{\gamma e}_{vf} &= 0, \\ \exists \xi_{ef} \text{ s.t. } G^\gamma_{ef} &= \exp(\xi_{ef} * \hat{B}^{\gamma e}_{vf}). \end{aligned}$$

And on the face we have

$$\begin{aligned} G^\gamma_f \triangleright B^v_{e'e} &= B^v_{e'e}, \\ \tilde{G}_f \triangleright B^v_{e'e} &= B^v_{e'e}. \\ G^\gamma_f &= ???. \end{aligned}$$

The geometric interpretation - Flatness

The untwisted G_{ev} and G_{ef} do not create a rotation in the plane of the triangle, but G^{γ}_{ef} do. $G^{\gamma}_{f} = \mathbf{1}$ sets this rotation to the identity. As the rotation in the plane is proportional to the rotation orthogonal to the plane, and the rotation orthogonal to the plane is the deficit rotation, the deficit angle is set to zero:

$$G^{\gamma}_{f} = \mathbf{1} \to \gamma\Theta = 0 \text{ mod } 2\pi$$

To allow for rotations on the plane of the triangle we need to modify the face identity as such:

The twisted face equation

$$G^{\gamma}_{f} = (\gamma *) \triangleright \tilde{G}_{f}.$$

The partition function - Fixing flatness

Thus $\delta(g_f) \to D_\gamma(g_f, \tilde{g}_f)$:

The corrected Holonomy Spin Foam Partition Function

$$\mathcal{Z}_\gamma(\mathcal{C}) = \int \left(\prod_{e \subset f} \mathrm{d}g_{ef} \right) \left(\prod_{v \subset e} \mathrm{d}g_{ev} \right) \left(\prod_{e \subset f} E_\gamma(g_{ef}) \right) \left(\prod_{f} D_\gamma(g_f, \tilde{g}_f) \right)$$

We have a proposal using FK like construction.
BUT: Appears almost certain we lose the LQG Hilbertspace[4].
Still: A new Hilbert space appears. $L\left(G^{\#(ve)}/(G^{\#v} \times H^{\#e})\right)$ universal to all $H \subset G$, all boundaries (without the usual topological restrictions).

[4] I never promised you a rose garden.

The partition function - Regularisation

Most of the above is “if I decide to ignore that this isn't well defined”. The wave front set allows us to study how to make it well defined (wip). The problem are configurations for which p can become arbitrary large in the interior while remaining finite on the boundary:

BF: The theory contains an unconstrained $SU(2)$ sector. Thus regularizing the above theory is at least as hard as regularizing $\mathrm{SU}(2)$ BF theory. We know from Bonzom and Smerlak that this is not possible on the 2-complex alone. Construction ala Bahr might work for the observables though.

Diffeos: If we have a geometric flat configuration in the interior, with an internal vertex, we have vertex translation symmetry that allows us to move this vertex arbitrarily far out.

Are there more? What ambiguities occur when extending the distribution to these configurations? If the connection g_{ev} stays geometric, the flatness persists!

Conclusions

What's new:

1. First analysis of the partition function without assumptions on the interior.
2. Makes some generalizations obvious that are hard to see in the spin picture.
3. New boundary Hilbert space that is common to all models based on groups G, H, and allows gluings on arbitrary graphs, and study of transfer operator.

ToDo:

1. Lorentzian (fate of flatness).
2. Spinor formulation should be natural (WF sets play nice with symplectic reduction).
3. Study other simplicity constraints (holomorphic).
4. Study extension and it's ambiguities.
5. Take continuum limit!

Thank You!

The one issue which still needs to be addressed in order for the state sum we are proposing to be a candidate for a quantum theory of gravity is how probability amplitudes computed with it behave as we refine the triangulation.

– Barrett and Crane in '98