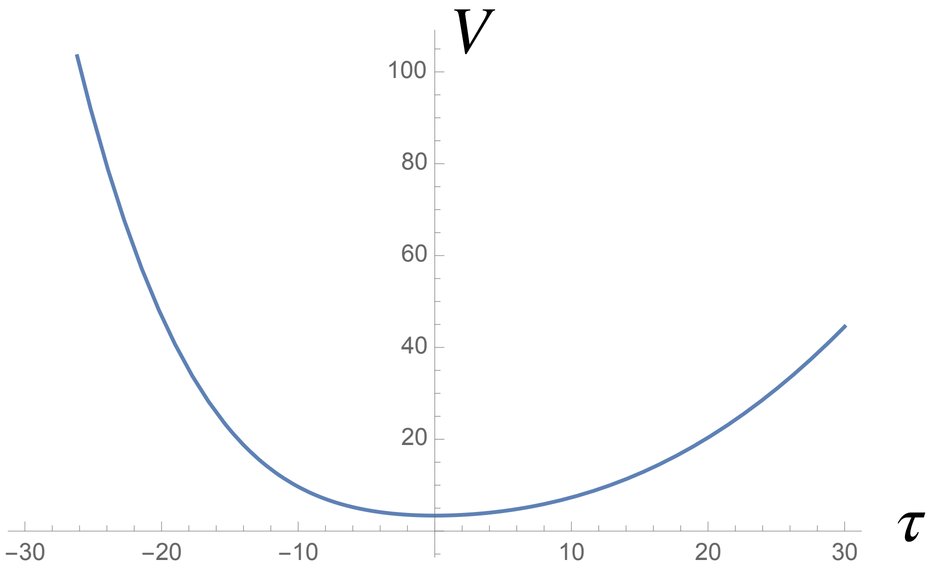
Effective Dynamics from Full Loop Quantum Gravity



Muxin Han

ILQGS Oct 2109

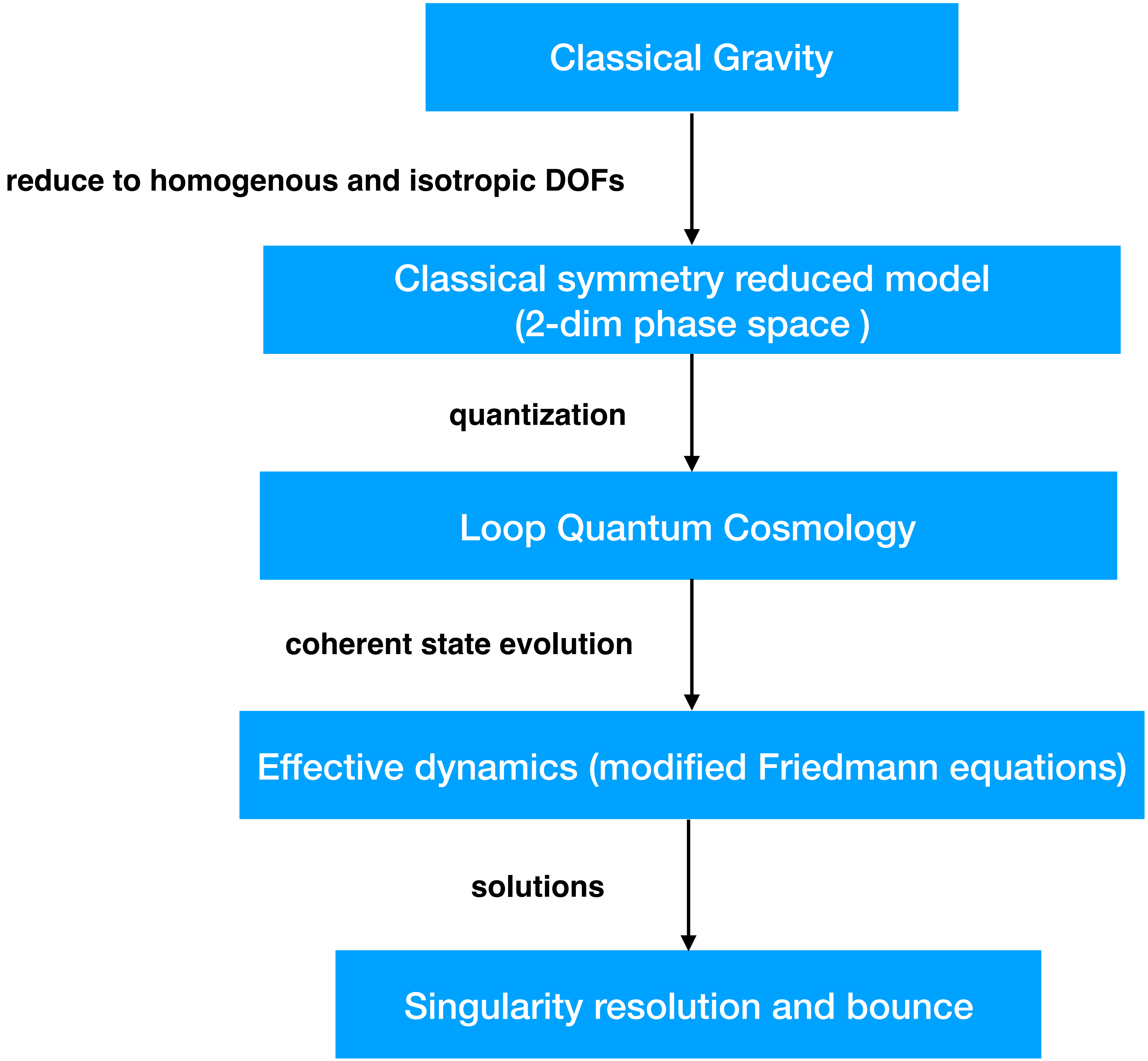
MH and Hongguang Liu, arXiv:1910.xxxxx

FAU FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

NSF

FAU FLORIDA ATLANTIC UNIVERSITY

Loop Quantum Cosmology (LQC)



Issues of LQC / Our motivations

1. How to relate LQC and singularity resolution to the full Loop Quantum Gravity?

2. Quantum fluctuation beyond the homogeneous and isotropic sector?

Recent interesting result

Recent work in Andrea Dapor and Klaus Liegener ’17:

Taking full LQG Hamiltonian and its coherent state expectation value at the homogeneous and isotropic data

$$\left\langle \Psi^t_{(C,P)} \middle| \hat{H}_{LQG} \middle| \Psi^t_{(C,P)} \right\rangle = -\frac{3}{8\pi G\beta^2\mu^2}\sqrt{P}\left[\sin^2(\mu C) - \left(1+\beta^2\right)\sin^4(\mu C) + \mathcal{O}(t)\right]$$

The result is viewed as an effective Hamiltonian of LQC, reproducing the Hamiltonian in You Ding, Yongge Ma, and Jinsong Yang ’09.

The effective dynamics = the classical evolution generated by this effective Hamiltonian —> unsymmetric bounce

Relation to the quantum dynamics of full LQG? It relies on the conjecture on the existence of dynamically stable coherent state in full LQG.

Our idea: Path integral formula

If we formulate the full LQG as path integral,

$$\int D\phi \, e^{\frac{i}{\hbar}S[\phi]}, \quad \hbar \to 0 \quad \Rightarrow \quad \delta S = 0$$

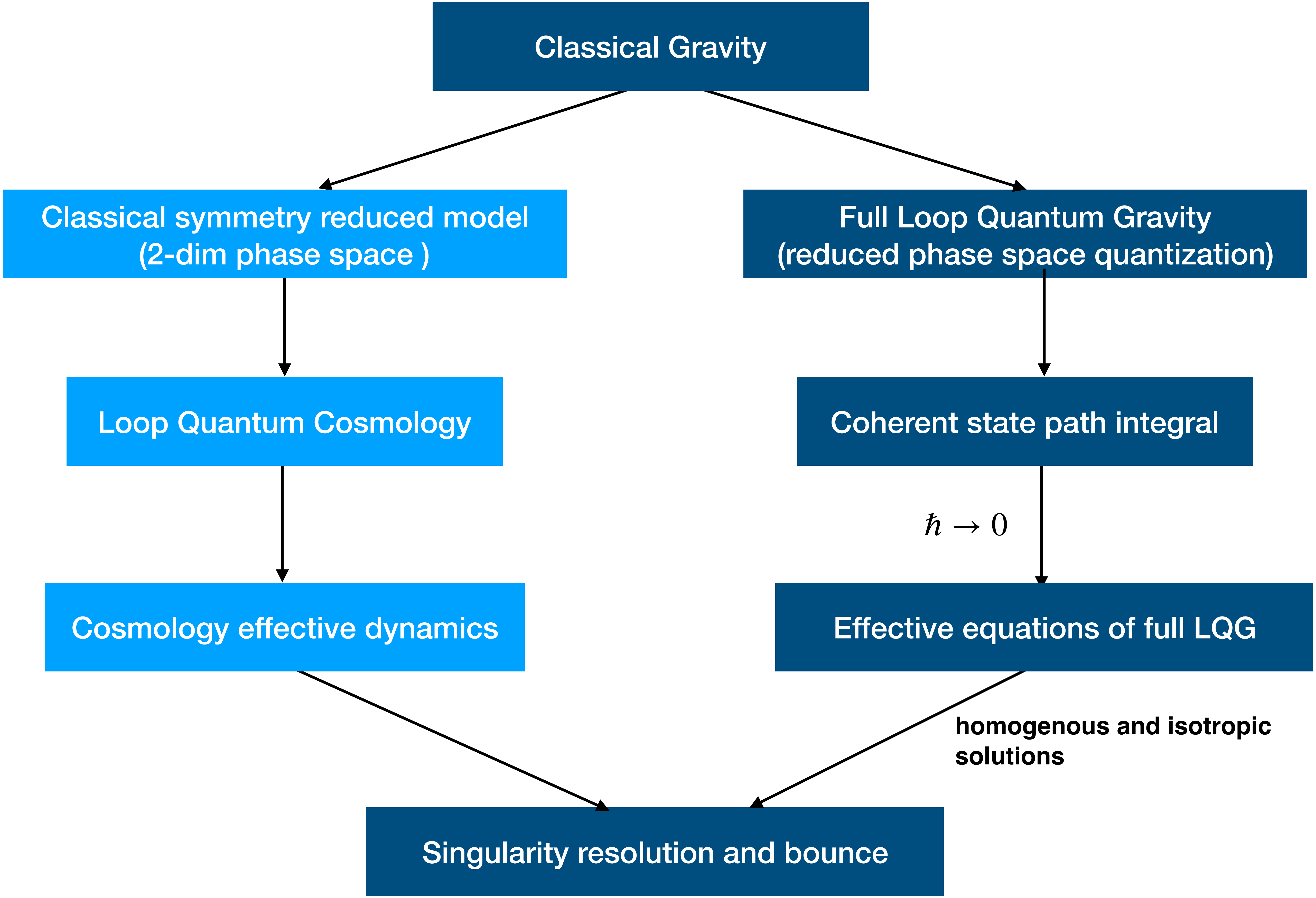
Our proposal:

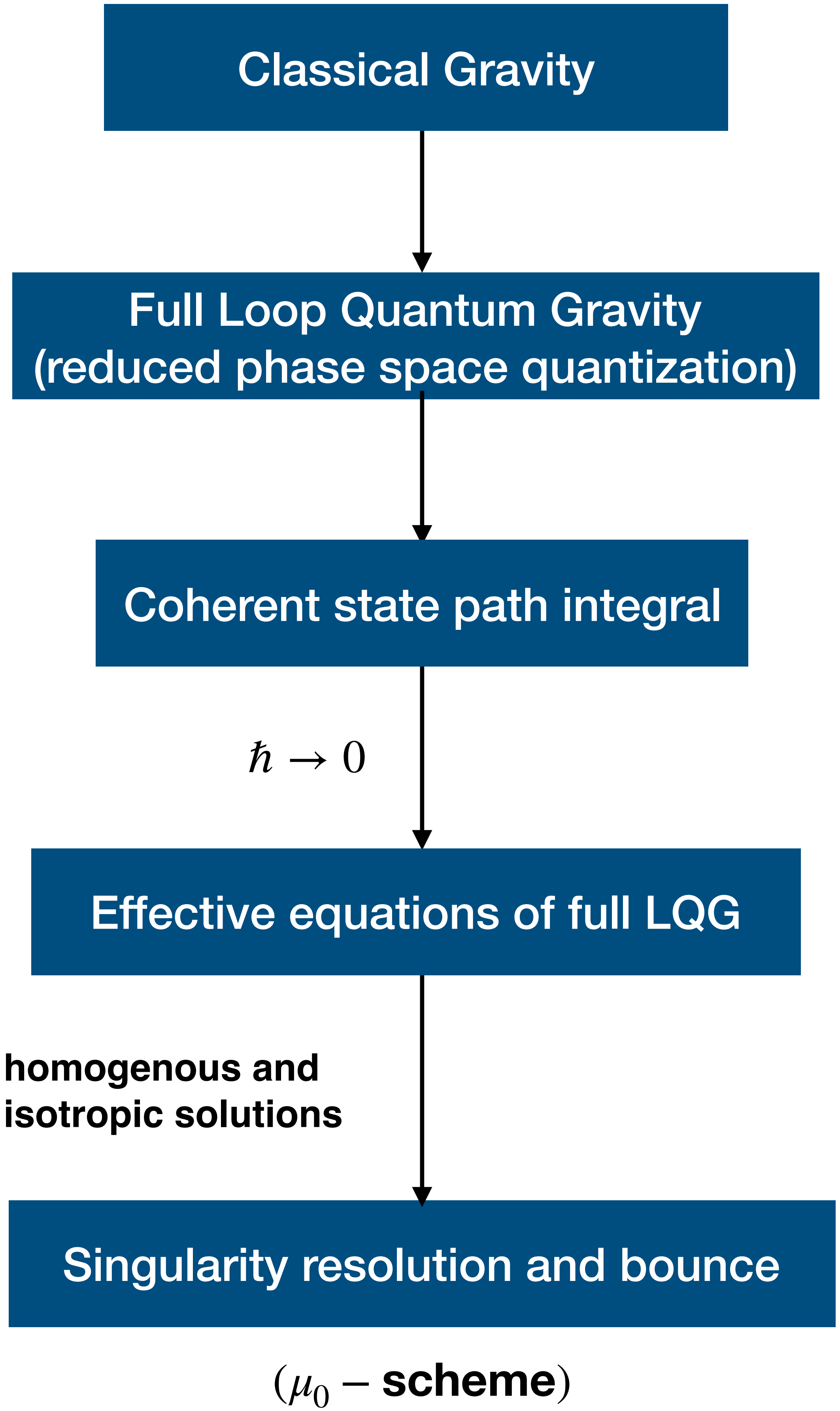
Cosmological Effective Dynamics = Solutions of Classical / Quantum Equations of Motion (EOMs) for full LQG

It doesn't rely on dynamically stable coherent state: lesson from interacting QFT

Given any solution of classical EOM, we can in principle compute all quantum fluctuations by standard perturbative expansion.

Path integral in LQC and relation to effective dynamics: Ashtekar, Campiglia, and Henderson, 2009, Henderson, Rovelli, Vidotto, and Wilson-Ewing, 2009, Qin and Ma, 2012, Craig and Singh, 2012


Classical Gravity
Classical symmetry reduced model
(2-dim phase space)
Full Loop Quantum Gravity
(reduced phase space quantization)
Loop Quantum Cosmology
Coherent state path integral
$\hbar \to 0$
Cosmology effective dynamics
Effective equations of full LQG
homogenous and isotropic
solutions
Singularity resolution and bounce



Some remarks

3 scenarios of deparametrized models: gravity coupled to

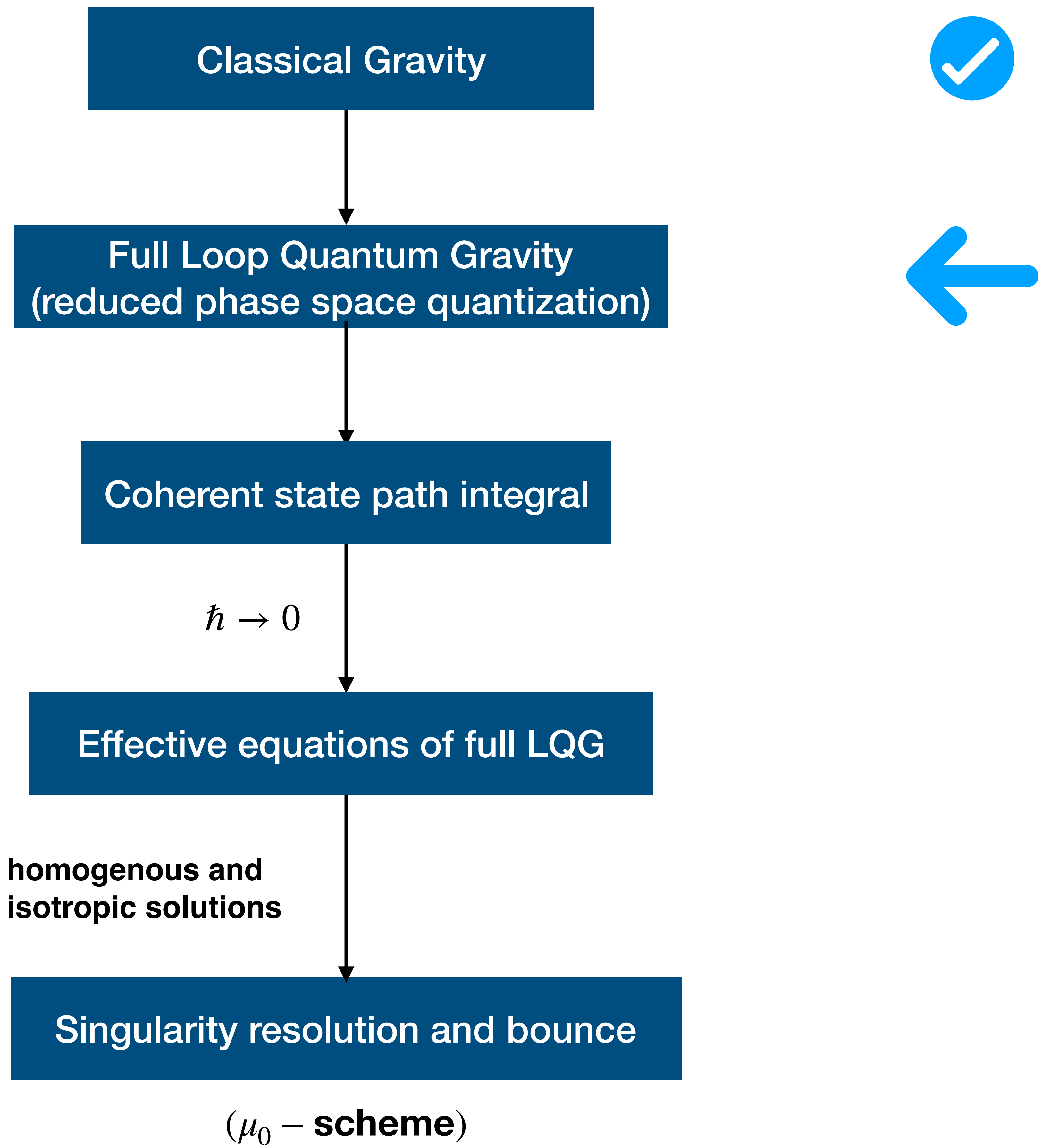
- **Brown-Kuchar dust**
- **Gaussian dust**
- **a massless scalar field**

The quantization is on a fixed graph γ and uses non-graph-changing Hamiltonian (γ is a finite cubic lattice partitioning 3-torus)

2 possible choices of physical Hamiltonian:

- **Giesel-Thiemann's Hamiltonian**
- **Alesci-Assanioussi-Lewandowski-Makinen's Hamiltonian (Warsaw's Hamiltonian)**

Our studies exhaust all $3 \times 2 = 6$ possible scenarios.


Classical Gravity
Full Loop Quantum Gravity
(reduced phase space quantization)
Coherent state path integral
$\hbar \to 0$
Effective equations of full LQG
homogenous and
isotropic solutions
Singularity resolution and bounce
(μ_0 − **scheme**)

Reduced phase space quantization

3 scenarios of deparametrized models: gravity coupled to

Brown and Kuchar 1994
Giesel and Thiemann 2007

- **Brown-Kuchar dust**

$$S_{BKD}\left[\rho, g_{\mu\nu}, T, S^j, W_j\right] = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{|\det(g)|}\,\rho\left[g^{\mu\nu}U_\mu U_\nu + 1\right], \quad U_\mu = -\partial_\mu T + W_j \partial_\mu S^j$$

Dirac observables = parametrizing gravity variables with values of dust fields

$$T(x) \equiv \tau, \quad S^j(x) \equiv \sigma^j$$

τ: physical time variable

σ: physical space variable

Rovelli 2001
Dittrich 2004
Thiemann 2004

Gravity Dirac observables

$$A(\sigma,\tau) = A(x)\Big|_{T(x)\equiv\tau, S^j(x)\equiv\sigma^j}, \quad E(\sigma,\tau) = E(x)\Big|_{T(x)\equiv\tau, S^j(x)\equiv\sigma^j}$$

- **Gaussian dust**

Kuchar and Torre 1990
Giesel and Thiemann 2015

$$S_{GD}\left[\rho, g_{\mu\nu}, T, S^j, W_j\right] = -\int \mathrm{d}^4x\sqrt{|\det(g)|}\left[\frac{\rho}{2}\left(g^{\mu\nu}\partial_\mu T\partial_\nu T + 1\right) + g^{\mu\nu}\partial_\mu T\left(W_j \partial_\nu S^j\right)\right]$$

- **a massless scalar field**

Rovelli and Smolin 1993
Domagala, Giesel, Kaminski, and Lewandowski 2010

$$S_\phi\left[g_{\mu\nu}, \phi\right] = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{|\det(g)|}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

Single ϕ only deparametrizes time: $\phi(x) \equiv \tau$ **τ: physical time variable**

$$A(\vec{x},\tau) = A(x)\Big|_{\phi(x)\equiv\tau}, \quad E(\vec{x},\tau) = E(x)\Big|_{\phi(x)\equiv\tau}$$

Reduced phase space quantization

Canonical structure of Dirac observables:

$$\left\{E_a^i(\sigma,\tau), A_j^b(\sigma',\tau)\right\} = \frac{1}{2}\kappa\beta\delta_j^i\delta_a^b\delta^3(\sigma,\sigma')$$

a,b,c…: SU(2) indices

i,j,k,…: spatial indices of the dust space $\mathcal{S}$

(space of $\boldsymbol{\sigma}$'s, slice with constant $\boldsymbol{\tau}$)

$$\kappa = 16\pi G$$

Solving constraints (Abelianized constraints):

$$C^{tot} = P + h\left(p, q, \partial_\alpha T\right) \approx 0, \quad C_j^{tot} = P_j + S_j^\alpha\left[C_\alpha(p,q) + P\partial_\alpha T\right] \approx 0$$

P : momentum of T

P_j : momentum of S_j

Physical Hamiltonian (generating $\boldsymbol{\tau}$ evolution):

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \{\mathbf{H}, f\}$$

- **Brown-Kuchar dust** Giesel and Thiemann 2007

$$\mathbf{H} = \int_{\mathcal{S}} \mathrm{d}^3\sigma \sqrt{C(\sigma,\tau)^2 - \frac{1}{4}\sum_{j=1}^{3} C_j(\sigma,\tau)C_j(\sigma,\tau)}$$

- **Gaussian dust** Giesel and Thiemann 2015

$$\mathbf{H} = \int_{\mathcal{S}} \mathrm{d}^3\sigma\ C(\sigma,\tau)$$

$$C = -\frac{2}{\kappa\sqrt{\det(q)}}\operatorname{tr}\left(F_{jk}\left[E^j, E^k\right]\right) + \frac{2\left(1+\beta^2\right)}{\kappa\sqrt{\det(q)}}\operatorname{tr}\left(\left[K_j, K_k\right]\left[E^j, E^k\right]\right)$$

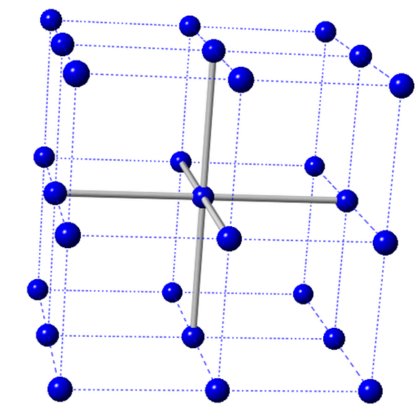
$$C_j = -\frac{2}{\kappa\sqrt{\det(q)}}\operatorname{tr}\left(\tau_j F_{kl}\left[E^k, E^l\right]\right)$$

- **a massless scalar field** Domagala, Giesel, Kaminski, and Lewandowski 2010, Rovelli and Smolin 1993

$$\mathbf{H} = \int_{\mathcal{S}} \mathrm{d}^3x \sqrt{-\sqrt{\det(q)}C + \sqrt{\det(q)}\sqrt{C^2 - q^{\alpha\beta}C_\alpha C_\beta}}$$

$C_\alpha \approx 0$ should be imposed

Reduced phase space quantization



The quantization is on a fixed graph γ with periodic boundary condition (γ is a finite cubic lattice partitioning 3-torus, $\mathcal{S} \simeq T^3$)

Holonomy and flux at every edge (Dirac observables)

$$h(e) := \mathscr{P}\exp\int_e A, \quad \textbf{and} \quad p^a(e) := -\frac{1}{2\beta a^2}\,\mathrm{tr}\left[\tau^a \int_{S_e} \varepsilon_{ijk}\mathrm{d}\sigma^i \wedge \mathrm{d}\sigma^j h\left(\rho_e(\sigma)\right) E_b^k(\sigma)\tau^b h\left(\rho_e(\sigma)\right)^{-1}\right]$$

$\tau^a = -i$(**Pauli matrix**)^a

$$\mathscr{H}_\gamma^0 = \otimes_e L^2(\mathrm{SU}(2)) \longrightarrow \mathscr{H}_\gamma$$

Gauss constraint is imposed quantum mechanically

$\mathscr{H}_\gamma$ **is already physical Hilbert space because it is constructed with Dirac observables**

Reduced phase space quantization

Non-graph-changing Hamiltonian: Positive and self-adjoint

- **Brown-Kuchar/Gaussian dust**

  $\alpha = 1$ **or** 0

$$\hat{\mathbf{H}} = \sum_{v \in V(\gamma)} \left[\hat{M}_-^\dagger(v)\hat{M}_-(v)\right]^{1/4} \qquad \hat{M}_-(v) = \hat{C}_v^\dagger \hat{C}_v - \frac{\alpha}{4}\hat{C}_{j,v}^\dagger \hat{C}_{j,v},$$

$$\hat{C}_{\mu,v} := -\frac{4}{3i\beta\kappa\ell_p^2/2} \sum_{s_1,s_2,s_3=\pm1} s_1 s_2 s_3 \varepsilon^{I_1 I_2 I_3} \mathrm{Tr}\left(\tau_\mu \hat{h}\left(\alpha_{v;I_1 s_1, I_2 s_2}\right) \hat{h}\left(e_{v;I_3 s_3}\right)\left[\hat{h}\left(e_{v;I_3 s_3}\right)^{-1}, \hat{V}_v\right]\right), \quad \mu = 0,1,2,3$$

Giesel-Thiemann's Hamiltonian

Thiemann 1996
Giesel and Thiemann 2006, 2007

$$\hat{C}_v = \hat{C}_{0,v} + \frac{1+\beta^2}{2}\hat{C}_{L,v}, \quad \hat{K} = \frac{i}{\hbar\beta^2}\left[\sum_{v\in V(\gamma)} \hat{C}_{0,v}, \sum_{v \in V(\gamma)} V_v\right]$$

$$\hat{C}_{L,v} = \frac{16}{3\kappa\left(i\beta\ell_p^2/2\right)^3} \sum_{s_1,s_2,s_3=\pm1} s_1 s_2 s_3 \varepsilon^{l_1 l_2 l_3}\, \mathrm{Tr}\left(\hat{h}\left(e_{v;I_1 s_1}\right)\left[\hat{h}\left(e_{v;h_1 s_1}\right)^{-1}, \hat{K}\right]\hat{h}\left(e_{v;l_2 s_2}\right)\left[\hat{h}\left(e_{v;i_2 s_2}\right)^{-1}, \hat{K}\right]\hat{h}\left(e_{v;I_3 s_3}\right)\left[\hat{h}\left(e_{v;I_3 s_3}\right)^{-1}, \hat{V}_v\right]\right)$$

Warsaw's Hamiltonian

Alesci, Assanioussi, Lewandowski, and Makinen 2015

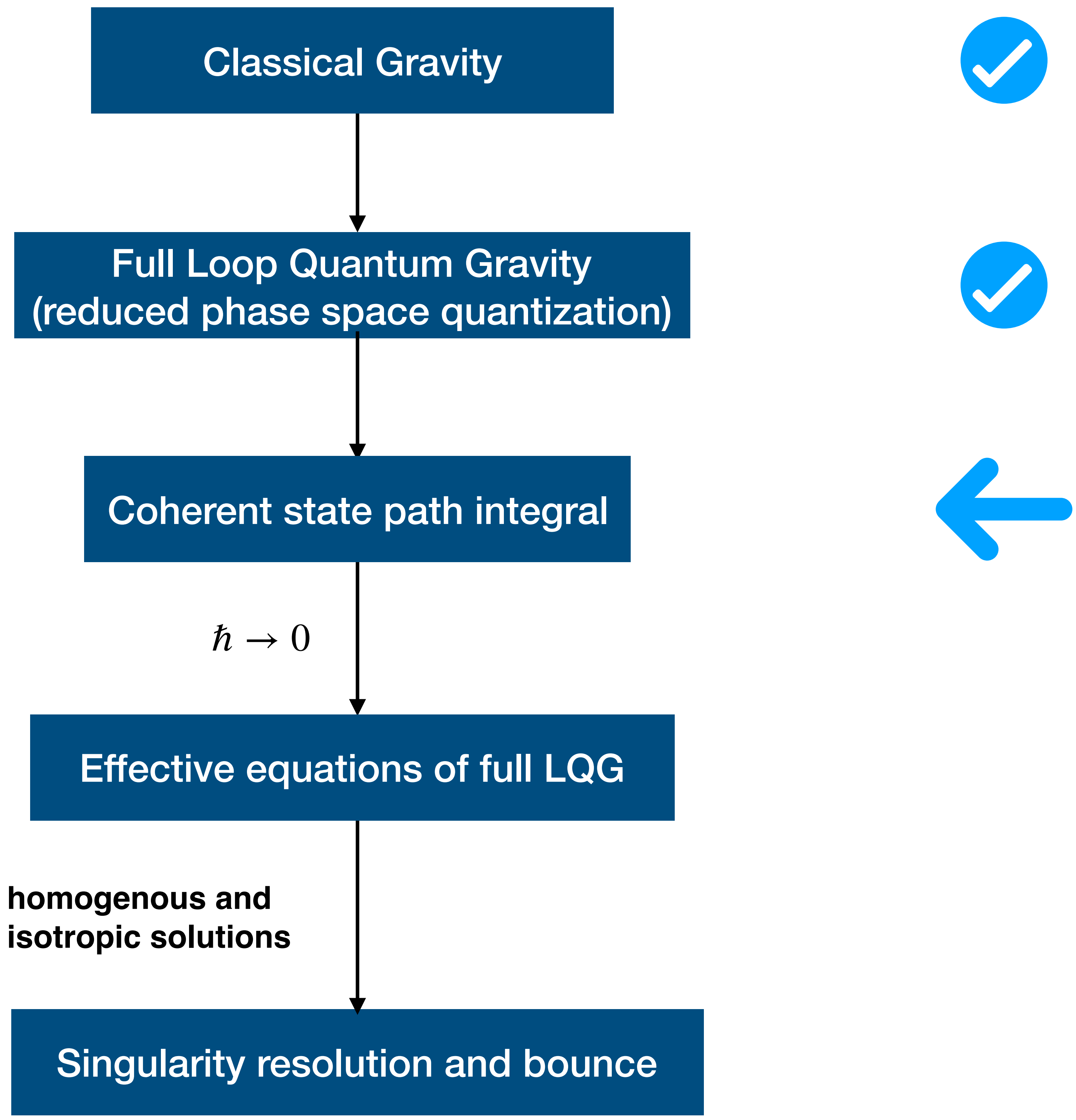
$$\hat{C}_v = -\frac{1}{\beta^2}\hat{C}_{0,v} - \frac{1+\beta^2}{\kappa\beta^2}{}^3\hat{R}_v \longleftarrow$$ **scalar curvature operator**

- **a massless scalar field** $\quad \mathscr{H}_\gamma \longrightarrow \mathscr{H}_{\gamma,Diff}$

Rovelli and Smolin 1993
Domagala, Giesel, Kaminski, and Lewandowski 2010

$$\hat{\mathbf{H}} = \sum_{v\in V(\gamma)} \left(\hat{V}_v \hat{C}_v^\dagger \hat{C}_v \hat{V}_v\right)^{1/4}$$

defined on $\mathscr{H}_{\gamma,Diff}$


Classical Gravity
Full Loop Quantum Gravity
(reduced phase space quantization)
Coherent state path integral
$\hbar \to 0$
Effective equations of full LQG
homogenous and
isotropic solutions
Singularity resolution and bounce

Complexifier Coherent States

At one edge: $$\psi^t_{g(e)}\left(h(e)\right) = \sum_{j_e}\left(2j_e+1\right)e^{-tj_e(j_e+1)/2}\chi_{j_e}\left(g(e)h(e)^{-1}\right)$$ Sahlmann, Thiemann, Winkler 2000 - 2001

Coherent state label: $g(e) = e^{-ip^a(e)\tau^a/2}e^{\theta^a(e)\tau^a/2} \in \mathrm{SL}(2,\mathbb{C}), \quad p_a(e), \theta_a(e) \in \mathbb{R}^3$ **complexified holonomy**

Holomorphic parametrization of LQG phase space

$p^a(e)$: **flux**

$e^{\theta^a(e)\tau^a/2}$: **holonomy**

Semiclassicality parameter (dimensionless): $t = \dfrac{\ell_P^2}{a^2}$, a **is a length unit, e.g. 1cm**, $t \to 0$ $\quad (\ell_p^2 = \hbar\kappa)$

Normalized coherent state $$\tilde{\psi}^t_{g(e)} = \frac{\psi^t_{g(e)}}{\left\| \psi^t_{g(e)} \right\|}$$

Overcompleteness: $$\int_{G^{\mathbb{C}}} \mathrm{d}g(e) \left|\tilde{\psi}^t_{g(e)}\right\rangle\left\langle\tilde{\psi}^t_{g(e)}\right| = 1_{\mathscr{H}_e}, \quad \mathrm{d}g(e) = \frac{c}{t^3}\mathrm{d}\mu_H(h(e))\mathrm{d}^3p(e), \quad c = \frac{2}{\pi} + o\left(t^\infty\right)$$

On a graph γ : $$\psi^t_g = \bigotimes_{e\in E(\gamma)} \psi^t_{g(e)} \qquad \tilde{\psi}^t_g = \bigotimes_{e\in E(\gamma)} \tilde{\psi}^t_{g(e)}$$

Gauge invariant coherent state:
(labelled by gauge orbit $[g]$)

$$\Psi^t_{[g]} = \int \mathrm{d}h\, \psi^t_{g^h}, \quad \text{where} \quad g^h = \left\{h^{-1}_{s(e)}g(e)h_{t(e)}\right\}_{e\in E(\gamma)}, \quad \mathrm{d}h = \prod_{v\in V(\gamma)} \mathrm{d}\mu_H\left(h_v\right)$$

Coherent States Path Integral

Given any non-graph-changing, positive, and self-adjoint physical Hamiltonian $\hat{\mathbf{H}}$

Transition amplitude between 2 gauge invariant coherent states:

$$A_{[g],[g']} := \left\langle \Psi^t_{[g]} \middle| U(T) \middle| \Psi^t_{[g']} \right\rangle_{\mathscr{H}_\gamma}, \quad U(\tau) := \exp\left(-\frac{i}{\hbar} T \hat{\mathbf{H}}\right)$$

$$= \int \mathrm{d}h \left\langle \psi^t_g \middle| U(T) \middle| \psi^t_{g'^h} \right\rangle$$

Additional diffeomorphism average for gravity-scalar model

Discretization and insert N+1 overcompleteness relations

$$\left\langle \psi^t_g \middle| U(T) \middle| \psi^t_{g'^h} \right\rangle = \left\langle \psi^t_g \middle| \left[e^{-\frac{i}{\hbar}\Delta\tau\hat{\mathbf{H}}} \right]^N \middle| \psi^t_{g'^h} \right\rangle,$$ **where** $\Delta\tau = T/N$ **arbitrarily small,**

$$= \int \mathrm{d}g_{N+1} \cdots \mathrm{d}g_1 \langle \psi^t_g | \tilde{\psi}^t_{g_{N+1}} \rangle \langle \tilde{\psi}^t_{g_{N+1}} \middle| e^{-\frac{i}{\hbar}\Delta\tau\hat{\mathbf{H}}} \middle| \tilde{\psi}^t_{g_N} \rangle \langle \tilde{\psi}^t_{g_N} \middle| e^{-\frac{i}{\hbar}\Delta\tau\hat{\mathbf{H}}} \middle| \tilde{\psi}^t_{g_{N-1}} \rangle \cdots \langle \tilde{\psi}^t_{g_2} \middle| e^{-\frac{i}{\hbar}\Delta\tau\hat{\mathbf{H}}} \middle| \tilde{\psi}^t_{g_1} \rangle \langle \tilde{\psi}^t_{g_1} | \psi^t_{g'^h} \rangle$$

Because $U(\Delta\tau)$ is a strongly continuous unitary group,

$$\left\langle \tilde{\psi}^t_{g_{i+1}} \middle| \exp\left(-\frac{i}{\hbar}\Delta\tau\hat{\mathbf{H}}\right) \middle| \tilde{\psi}^t_{g_i} \right\rangle = \left\langle \tilde{\psi}^t_{g_{i+1}} \middle| 1 - \frac{i\Delta\tau}{\hbar}\hat{\mathbf{H}} \middle| \tilde{\psi}^t_{g_i} \right\rangle + \frac{\Delta\tau}{\hbar}\varepsilon_{i+1,i}\left(\frac{\Delta\tau}{\hbar}\right)$$

$$= \left\langle \tilde{\psi}^t_{g_{i+1}} \middle| \tilde{\psi}^t_{g_i} \right\rangle e^{-\frac{i\Delta\tau}{\hbar}\frac{\langle \psi^t_{g_{i+1}} | \hat{\mathbf{H}} | \psi^t_{g_i} \rangle}{\langle \psi^t_{g_{i+1}} | \psi^t_{g_i} \rangle} + \frac{\Delta\tau}{\hbar}\epsilon_{i+1,i}\left(\frac{\Delta\tau}{\hbar}\right)}$$

Coherent States Path Integral

$$\left\langle \tilde{\psi}^t_{g_{i+1}} \middle| \exp\left(-\frac{i}{\hbar}\Delta\tau\hat{\mathbf{H}}\right) \middle| \tilde{\psi}^t_{g_i}\right\rangle = \left\langle \tilde{\psi}^t_{g_{i+1}} \middle| 1 - \frac{i\Delta\tau}{\hbar}\hat{\mathbf{H}} \middle| \tilde{\psi}^t_{g_i}\right\rangle + \frac{\Delta\tau}{\hbar}\varepsilon_{i+1,i}\left(\frac{\Delta\tau}{\hbar}\right)$$

$$= \left\langle \tilde{\psi}^t_{g_{i+1}} \middle| \tilde{\psi}^t_{g_i}\right\rangle e^{-\frac{i\Delta\tau}{\hbar}\frac{\langle\psi^t_{g_{i+1}}|\hat{\mathbf{H}}|\psi^t_{g_i}\rangle}{\langle\psi^t_{g_{i+1}}|\psi^t_{g_i}\rangle}+\frac{\Delta\tau}{\hbar}\epsilon_{i+1,i}\left(\frac{\Delta\tau}{\hbar}\right)}$$

Because $U(\Delta\tau)$ is a strongly continuous unitary group,

$$\hat{\varepsilon}\left(\frac{\Delta\tau}{\hbar}\right) := \frac{\hbar}{\Delta\tau}\left[U(\Delta\tau) - 1 + \frac{i\Delta\tau}{\hbar}\hat{\mathbf{H}}\right], \qquad \varepsilon_{i+1,i}\left(\frac{\Delta\tau}{\hbar}\right) = \left\langle \tilde{\psi}^t_{g_{i+1}} \middle| \hat{\varepsilon}\left(\frac{\Delta\tau}{\hbar}\right) \middle| \tilde{\psi}^t_{g_i}\right\rangle \qquad \lim_{\Delta\tau\to 0}\varepsilon_{i+1,i}\left(\frac{\Delta\tau}{\hbar}\right) = 0$$

$$\frac{\Delta\tau}{\hbar}\epsilon_{i+1,i}\left(\frac{\Delta\tau}{\hbar}\right) = \ln\left[1 - \frac{i\Delta\tau}{\hbar}\frac{\langle\psi^t_{g_{i+1}}|\hat{\mathbf{H}}|\psi^t_{g_i}\rangle}{\langle\psi^t_{g_{i+1}}|\psi^t_{g_i}\rangle} + \frac{\Delta\tau}{\hbar}\frac{\varepsilon_{i+1,i}\left(\Delta\tau/\hbar\right)}{\langle\tilde{\psi}^t_{g_{i+1}}|\tilde{\psi}^t_{g_i}\rangle}\right] + \frac{i\Delta\tau}{\hbar}\frac{\langle\psi^t_{g_{i+1}}|\hat{\mathbf{H}}|\psi^t_{g_i}\rangle}{\langle\psi^t_{g_{i+1}}|\psi^t_{g_i}\rangle} \qquad \lim_{\Delta\tau\to 0}\epsilon_{i+1,i}\left(\frac{\Delta\tau}{\hbar}\right) = 0$$

Overlap inner product between 2 normalized coherent states

$$\langle\tilde{\psi}^t_{g_2(e)}|\tilde{\psi}^t_{g_1(e)}\rangle = \left[\frac{\sinh\left(p_1(e)\right)\sinh\left(p_2(e)\right)}{p_1(e)p_2(e)}\right]^{1/2}\frac{z_{21}(e)}{\sinh(z_{21}(e))}e^{K\left(g_2(e),\,g_1(e)\right)/t}\left[1 + O(t^\infty)\right]$$

$$K\left(g_2(e), g_1(e)\right) = z_{21}(e)^2 - \frac{1}{2}p_2(e)^2 - \frac{1}{2}p_1(e)^2, \quad z_{21}(e) = \operatorname{arccosh}\left(x_{21}(e)\right), \quad x_{21}(e) = \frac{1}{2}\mathrm{Tr}\left[g_2(e)^\dagger g_1(e)\right]$$

Coherent States Path Integral

Discrete path integral

$$A_{[g],[g']} = ||\psi_g^t||\,||\psi_{g'}^t|| \int dh \prod_{i=1}^{N+1} dg_i \, \nu[g] \; \exp\left(\frac{S[g,h]}{t}\right)$$

$$t = \frac{\ell_P^2}{a^2} = \frac{\hbar\kappa}{a^2}$$

Effective action

$$S[g] = \sum_{i=0}^{N+1} \sum_{e \in E(\gamma)} \left[z_{i+1,i}(e)^2 - \frac{1}{2} p_{i+1}(e)^2 - \frac{1}{2} p_i(e)^2 \right] - \frac{i\kappa}{a^2} \sum_{i=1}^{N} \Delta\tau \left[\frac{\langle \psi_{g_{i+1}}^t | \hat{\mathbf{H}} | \psi_{g_i}^t \rangle}{\langle \psi_{g_{i+1}}^t | \psi_{g_i}^t \rangle} + i\epsilon_{i+1,i}\left(\frac{\Delta\tau}{\hbar}\right) \right]$$

$$z_{i+1,i}(e) = \operatorname{arccosh}\left(x_{i+1,i}(e)\right), \quad x_{i+1,i}(e) = \frac{1}{2} \mathrm{Tr}\left[g_{i+1}(e)^\dagger g_i(e) \right]$$

negligible when $\Delta\tau \to 0$

Measure factor (independent of t)

$$\nu[g] = \prod_{i=1}^{N+2} \left\{ \prod_{e \in E(\gamma)} \left[\frac{\sinh(p_i(e))}{p_i(e)} \frac{\sinh(p_{i-1}(e))}{p_{i-1}(e)} \right]^{1/2} \frac{z_{i,i-1}(e)}{\sinh(z_{i,i-1}(e))} \right\}$$

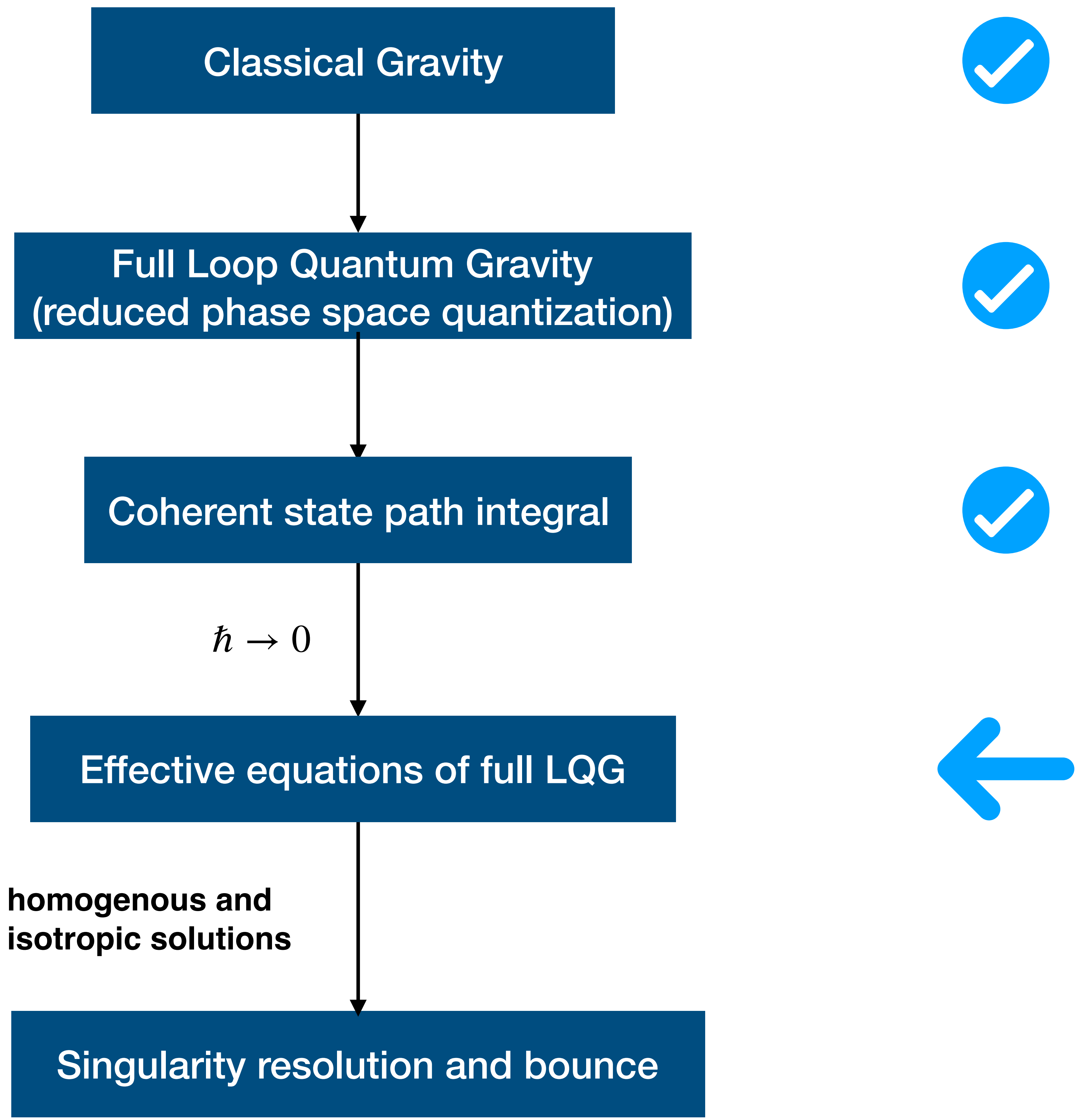
$$g(e) = e^{-ip^a(e)\tau^a/2} e^{\theta^a(e)\tau^a/2}$$

As $\Delta\tau \to 0$, the path integral is dominated by $||g_{i+1} - g_i|| \sim O(\sqrt{t})$.

At every step,

$$\left\langle \tilde{\psi}_{g_{i+1}}^t \middle| \exp\left(-\frac{i}{\hbar}\Delta\tau\hat{\mathbf{H}}\right) \middle| \tilde{\psi}_{g_i}^t \right\rangle = \left\langle \tilde{\psi}_{g_{i+1}}^t \middle| \tilde{\psi}_{g_i}^t \right\rangle e^{-\frac{i\Delta\tau}{\hbar} \frac{\langle \psi_{g_{i+1}}^t | \hat{\mathbf{H}} | \psi_{g_i}^t \rangle}{\langle \psi_{g_{i+1}}^t | \psi_{g_i}^t \rangle} + \frac{\Delta\tau}{\hbar}\epsilon_{i+1,i}\left(\frac{\Delta\tau}{\hbar}\right)}$$

where the overlap function $\left| \left\langle \tilde{\psi}_{g_{i+1}}^t \middle| \tilde{\psi}_{g_i}^t \right\rangle \right| \sim e^{-\frac{(\Delta p)^2 + (\Delta\theta)^2}{t}}$ **behaves as a Gaussian**



Classical Gravity
Full Loop Quantum Gravity
(reduced phase space quantization)
Coherent state path integral
$\hbar \to 0$
Effective equations of full LQG
homogenous and
isotropic solutions
Singularity resolution and bounce

Equations of motion (EOMs)

$$A_{[g],[g']} = ||\psi^t_g||\,||\psi^t_{g'}|| \int dh \prod_{i=1}^{N+1} dg_i\, \nu[g] \;\exp\left(\frac{S[g,h]}{t}\right) \qquad t = \frac{\ell_P^2}{a^2} = \frac{\hbar\kappa}{a^2}$$

Semiclassical limit $\hbar \to 0$ or $t \to 0$ and stationary phase approximation

path integral is dominated by critical points satisfying EOM $\delta S[g,h] = 0$

Variations:

- **Holomorphic deformation in** $\mathrm{SL}(2,\mathbb{C})$ $\quad g_i(e) \mapsto g_i^\varepsilon(e) = g_i(e) e^{\varepsilon_i^a(e)\tau^a}, \qquad \varepsilon_i^a(e) \in \mathbb{C}$
- **Real deformation in SU(2)** $\quad h_v \mapsto h_v^\eta = h_v e^{\eta_v^a \tau^a} \qquad \eta_v^a \in \mathbb{R}$

$$\frac{\delta S}{\delta \varepsilon_i^a(e)} = 0: \quad \frac{z_{i+1,i}(e)\mathrm{Tr}\left[\tau^a g_{i+1}^\dagger(e) g_i(e)\right]}{\sqrt{x_{i+1,i}(e)-1}\sqrt{x_{i+1,i}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = \frac{i\kappa\Delta\tau}{a^2}\frac{\partial}{\partial\varepsilon_i^a(e)}\frac{\langle \psi^t_{g^\varepsilon_{i+1}}|\hat{\mathbf{H}}|\psi^t_{g^\varepsilon_i}\rangle}{\langle \psi^t_{g^\varepsilon_{i+1}}|\psi^t_{g^\varepsilon_i}\rangle}\Bigg|_{\varepsilon=0} \quad i = 1,\cdots,N$$

$$\frac{\delta S}{\delta \varepsilon_i^a(e)^*} = 0: \quad \frac{z_{i,i-1}(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_{i-1}(e)\right]}{\sqrt{x_{i,i-1}(e)-1}\sqrt{x_{i,i-1}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = -\frac{i\kappa\Delta\tau}{a^2}\frac{\partial}{\partial\varepsilon_i^a(e)^*}\frac{\langle \psi^t_{g^\varepsilon_i}|\hat{\mathbf{H}}|\psi^t_{g^\varepsilon_{i-1}}\rangle}{\langle \psi^t_{g^\varepsilon_i}|\psi^t_{g^\varepsilon_{i-1}}\rangle}\Bigg|_{\varepsilon=0}, \quad i = 2,\cdots,N+1$$

$$\frac{\delta S[g]}{\delta \eta_v^a} = 0: \quad \sum_{e,t(e)=v} \Lambda^a_{\ c}(\theta)\, p_0^c(e) - \sum_{e,s(e)=v} p_0^a(e) = 0 \qquad e^{\theta^a\tau^a/2}\tau^a e^{-\theta^a\tau^a/2} = \Lambda^a_{\ b}(\theta)\tau^b$$

closure constraint (of cube) on initial data

initial state: $g_0 = g'^h$ **final state:** $g_{N+2} = g$

Equations of motion (EOMs)

$$\frac{\delta S}{\delta \varepsilon_i^a(e)} = 0: \qquad \frac{z_{i+1,i}(e)\mathrm{Tr}\left[\tau^a g_{i+1}^\dagger(e) g_i(e)\right]}{\sqrt{x_{i+1,i}(e)-1}\sqrt{x_{i+1,i}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = \frac{i\kappa\Delta\tau}{a^2}\frac{\partial}{\partial \varepsilon_i^a(e)} \frac{\langle \psi^t_{g^\varepsilon_{i+1}} | \hat{\mathbf{H}} | \psi^t_{g^\varepsilon_i}\rangle}{\langle \psi^t_{g^\varepsilon_{i+1}} | \psi^t_{g^\varepsilon_i}\rangle}\Bigg|_{\varepsilon=0} \qquad i = 1,\cdots,N$$

$$\frac{\delta S}{\delta \varepsilon_i^a(e)^*} = 0: \qquad \frac{z_{i,i-1}(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_{i-1}(e)\right]}{\sqrt{x_{i,i-1}(e)-1}\sqrt{x_{i,i-1}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = -\frac{i\kappa\Delta\tau}{a^2}\frac{\partial}{\partial \varepsilon_i^a(e)^*} \frac{\langle \psi^t_{g^\varepsilon_i} | \hat{\mathbf{H}} | \psi^t_{g^\varepsilon_{i-1}}\rangle}{\langle \psi^t_{g^\varepsilon_i} | \psi^t_{g^\varepsilon_{i-1}}\rangle}\Bigg|_{\varepsilon=0}, \qquad i = 2,\cdots,N+1$$

$$z_{i+1,i}(e) = \operatorname{arccosh}\left(x_{i+1,i}(e)\right), \quad x_{i+1,i}(e) = \frac{1}{2}\mathrm{Tr}\left[g_{i+1}(e)^\dagger g_i(e)\right]$$

For any solution $\{g_i(e)\}_{i=1}^{N+1}$ of EOMs,

$\Delta\tau \to 0$ ⟶ **"left-hand sides —> 0"** ⟶ **In the neighborhood $||g_{i+1} - g_i|| \sim O(\sqrt{t})$,**

$$g_{i+1}(e) \to g_i(e)$$

i.e. solutions $g_i(e) \equiv g_\tau(e)$ are continuous in τ when $\Delta\tau \to 0$.

(continuous approximation of discrete solutions)

<u>**Lemma:**</u> $g_{i+1}(e) = g_i(e)$ ***and*** $g_i(e) = g_{i-1}(e)$ ***are isolated roots of "left-hand sides = 0"***

$$\frac{z_{i+1,i}(e)\mathrm{Tr}\left[\tau^a g_{i+1}^\dagger(e) g_i(e)\right]}{\sqrt{x_{i+1,i}(e)-1}\sqrt{x_{i+1,i}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = 0, \qquad \frac{z_{i,i-1}(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_{i-1}(e)\right]}{\sqrt{x_{i,i-1}(e)-1}\sqrt{x_{i,i-1}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = 0.$$

Equations of motion (EOMs)

$$\frac{\delta S}{\delta \varepsilon_i^a(e)} = 0: \qquad \frac{z_{i+1,i}(e)\mathrm{Tr}\left[\tau^a g_{i+1}^\dagger(e) g_i(e)\right]}{\sqrt{x_{i+1,i}(e)-1}\sqrt{x_{i+1,i}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = \frac{i\kappa\Delta\tau}{a^2}\frac{\partial}{\partial \varepsilon_i^a(e)}\frac{\langle \psi^t_{g^\varepsilon_{i+1}}|\hat{\mathbf{H}}|\psi^t_{g_i^\varepsilon}\rangle}{\langle \psi^t_{g^\varepsilon_{i+1}}|\psi^t_{g_i^\varepsilon}\rangle}\Bigg|_{\varepsilon=0} \qquad i = 1,\cdots,N$$

$$\frac{\delta S}{\delta \varepsilon_i^a(e)^*} = 0: \qquad \frac{z_{i,i-1}(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_{i-1}(e)\right]}{\sqrt{x_{i,i-1}(e)-1}\sqrt{x_{i,i-1}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = -\frac{i\kappa\Delta\tau}{a^2}\frac{\partial}{\partial \varepsilon_i^a(e)^*}\frac{\langle \psi^t_{g^\varepsilon_{i}}|\hat{\mathbf{H}}|\psi^t_{g_{i-1}^\varepsilon}\rangle}{\langle \psi^t_{g^\varepsilon_{i}}|\psi^t_{g_{i-1}^\varepsilon}\rangle}\Bigg|_{\varepsilon=0}, \qquad i = 2,\cdots,N+1$$

$$z_{i+1,i}(e) = \mathrm{arccosh}\left(x_{i+1,i}(e)\right), \quad x_{i+1,i}(e) = \frac{1}{2}\mathrm{Tr}\left[g_{i+1}(e)^\dagger g_i(e)\right]$$

Taking the continuous approximation, on the right hand side:

Lemma:

$$\lim_{g_i \to g_{i+1} \equiv g} \frac{\partial}{\partial \varepsilon_i^a(e)}\frac{\langle \psi^t_{g^\varepsilon_{i+1}}|\hat{\mathbf{H}}|\psi^t_{g_i^\varepsilon}\rangle}{\langle \psi^t_{g^\varepsilon_{i+1}}|\psi^t_{g_i^\varepsilon}\rangle}\Bigg|_{\vec{\varepsilon}=0} = \frac{\partial\langle \tilde{\psi}^t_{g^\varepsilon}|\hat{\mathbf{H}}|\tilde{\psi}^t_{g^\varepsilon}\rangle}{\partial \varepsilon^a(e)}\Bigg|_{\vec{\varepsilon}=0}$$

$$\lim_{g_{i-1} \to g_{i} \equiv g} \frac{\partial}{\partial \varepsilon_i^a(e)^*}\frac{\langle \psi^t_{g^\varepsilon_{i}}|\hat{\mathbf{H}}|\psi^t_{g_{i-1}^\varepsilon}\rangle}{\langle \psi^t_{g^{\varepsilon s}_{i}}|\psi^t_{g_{i-1}^\varepsilon}\rangle}\Bigg|_{\vec{\varepsilon}=0} = \frac{\partial\langle \tilde{\psi}^t_{g^\varepsilon}|\hat{\mathbf{H}}|\tilde{\psi}^t_{g^\varepsilon}\rangle}{\partial \varepsilon^a(e)^*}\Bigg|_{\vec{\varepsilon}=0}$$

reduces matrix element of $\hat{\mathbf{H}}$ (hard to compute) to expectation value of $\hat{\mathbf{H}}$ (easier to compute).

Semiclassical perturbation theory: $\langle \tilde{\psi}^t_{g^\varepsilon}|\hat{\mathbf{H}}|\tilde{\psi}^t_{g^\varepsilon}\rangle = \mathbf{H}\left[g^\varepsilon\right] + O(\hbar)$ **Giesel and Thiemann 2006**

Classical discrete Hamiltonian

Equations of motion (EOMs)

EOMs (at the leading order in $\hbar$) at every edge $e \in E(\gamma)$:

$$\frac{z_{i+1,i}(e)\mathrm{Tr}\left[\tau^a g_{i+1}^\dagger(e) g_i(e)\right]}{\sqrt{x_{i+1,i}(e)-1}\sqrt{x_{i+1,i}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = \frac{i\kappa\Delta\tau}{a^2}\frac{\partial \mathbf{H}[g_i^\varepsilon]}{\partial \varepsilon_i^a(e)}\Big|_{\varepsilon=0} \qquad i=1,\cdots,N$$

$$\frac{z_{i,i-1}(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_{i-1}(e)\right]}{\sqrt{x_{i,i-1}(e)-1}\sqrt{x_{i,i-1}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = -\frac{i\kappa\Delta\tau}{a^2}\frac{\partial \mathbf{H}[g^\varepsilon]}{\partial \varepsilon_i^a(e)^*}\Big|_{\varepsilon=0}, \qquad i=2,\cdots,N+1$$

$$z_{i+1,i}(e) = \mathrm{arccosh}\left(x_{i+1,i}(e)\right), \quad x_{i+1,i}(e) = \frac{1}{2}\mathrm{Tr}\left[g_{i+1}(e)^\dagger g_i(e)\right]$$

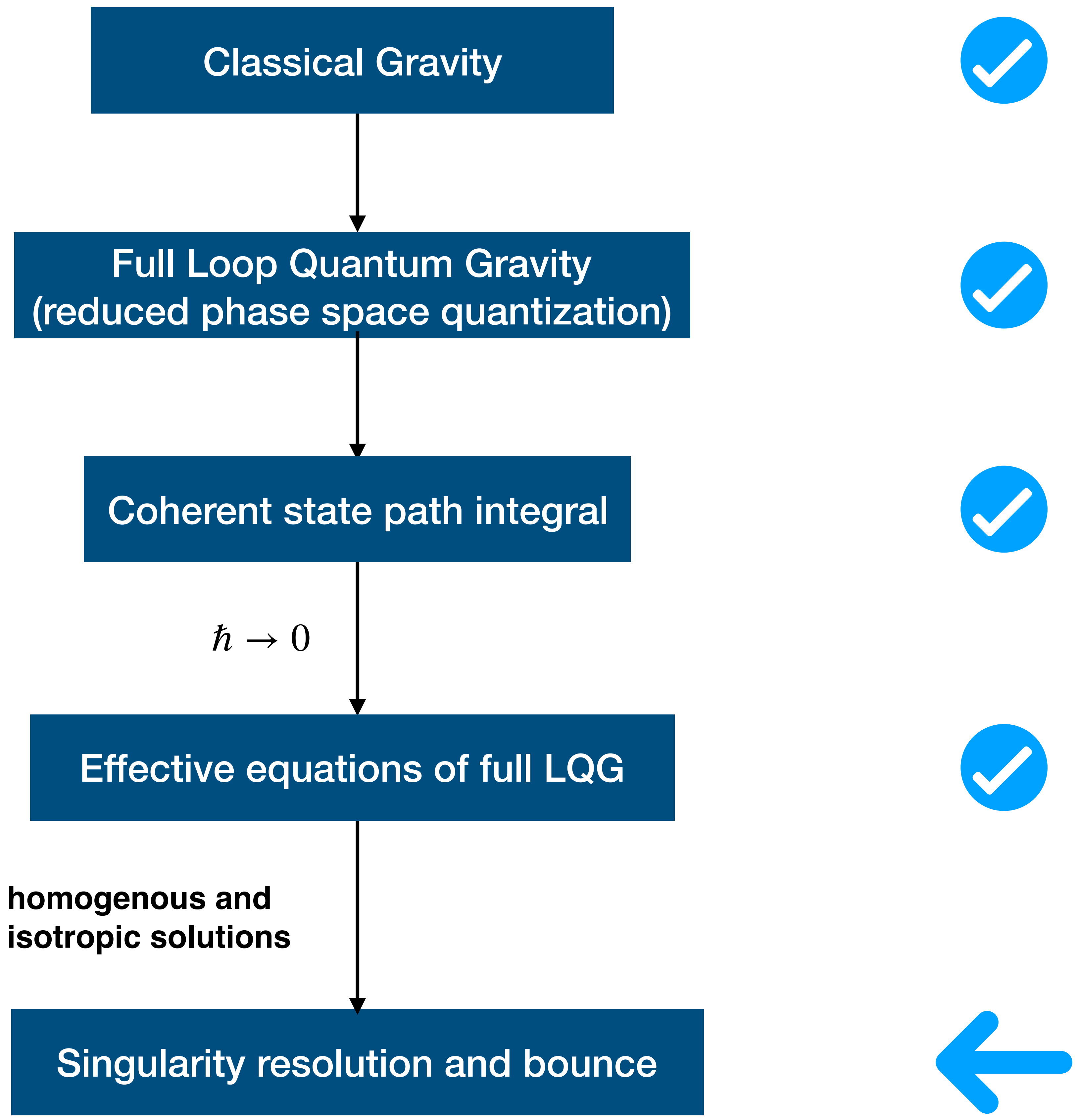
Closure constraint (of cube) at every vertex $v \in V(\gamma)$:

$$\sum_{e,t(e)=v} \Lambda^a{}_c(\theta)\, p_0^c(e) - \sum_{e,s(e)=v} p_0^a(e) = 0$$

initial condition: $g_0 = g'^h$ **final condition:** $g_{N+2} = g$

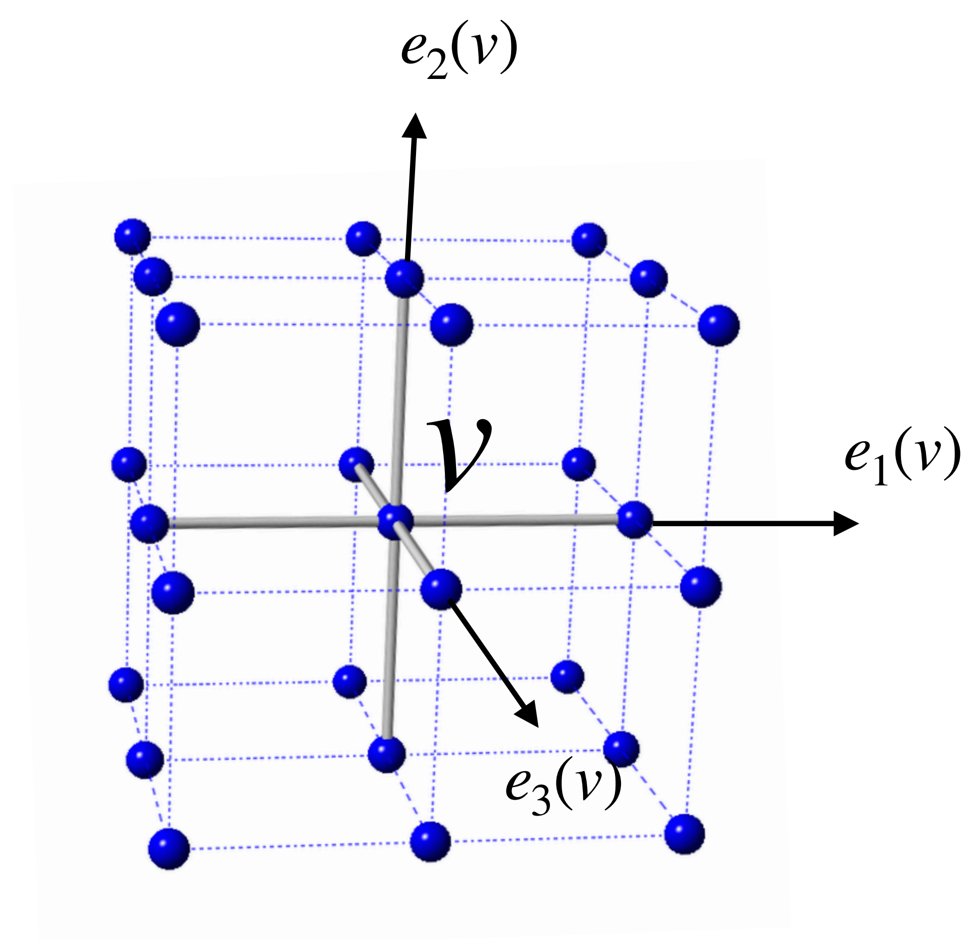
These are effective equations of the full LQG

- **Valid for all spacetimes,**
- **Solvable analytically or numerically,**
- **Quantum correction in principle can be computed by perturbative expansion of path integral.**



Classical Gravity
Full Loop Quantum Gravity
(reduced phase space quantization)
Coherent state path integral
$\hbar \to 0$
Effective equations of full LQG
homogenous and
isotropic solutions
Singularity resolution and bounce

Cosmological Solutions



Homogeneous and isotropic ansatz: ($I = 1,2,3$ label outgoing directions from v, $a = 1,2,3$ su(2) indices)

$$h_i\left(e_I(v)\right) = e^{\theta_i \tau_I/2}, \quad p_i^a\left(e_I(v)\right) = p_i\,\delta_I^a \qquad\qquad g_i\left(e_I(v)\right) = e^{(\theta_i - ip_i)\frac{\tau^I}{2}}$$

Dapor and Liegener 2017

p_i, θ_i **are cosmological variables (**i **labels time steps)**

closure constraint: $\sum_{e,t(e)=v} \Lambda^a{}_c(\theta)\, p_0^c(e) - \sum_{e,s(e)=v} p_0^a(e) = 0$ **is satisfied by the ansatz.**

Cosmological Solutions

Insert homogeneous and isotropic ansatz into EOMs

$$\frac{z_{i+1,i}(e)\mathrm{Tr}\left[\tau^a g_{i+1}^\dagger(e) g_i(e)\right]}{\sqrt{x_{i+1,i}(e)-1}\sqrt{x_{i+1,i}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = \frac{i\kappa\Delta\tau}{a^2}\frac{\partial \mathbf{H}[g_i^\varepsilon]}{\partial \varepsilon_i^a(e)}\bigg|_{\varepsilon=0} \qquad i = 1,\cdots,N$$

$$\frac{z_{i,i-1}(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_{i-1}(e)\right]}{\sqrt{x_{i,i-1}(e)-1}\sqrt{x_{i,i-1}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = -\frac{i\kappa\Delta\tau}{a^2}\frac{\partial \mathbf{H}[g^\varepsilon]}{\partial \varepsilon_i^a(e)^*}\bigg|_{\varepsilon=0}, \qquad i = 2,\cdots,N+1$$

↓

$$\delta_I^a\left[\frac{\theta_{i+1}-\theta_i}{\Delta\tau} + i\,\frac{p_{i+1}-p_i}{\Delta\tau}\right] = \frac{i\kappa}{a^2}\frac{\partial\,\mathbf{H}[g_i^\varepsilon]}{\partial\varepsilon_i^a\left(e_I(v)\right)}\bigg|_{\vec{\varepsilon}=0}, \qquad \delta_I^a\left[\frac{\theta_i-\theta_{i-1}}{\Delta\tau} - i\,\frac{p_i-p_{i-1}}{\Delta\tau}\right] = -\frac{i\kappa}{a^2}\frac{\partial\,\mathbf{H}[g_i^\varepsilon]}{\partial\varepsilon_i^a\left(e_I(v)\right)^*}\bigg|_{\vec{\varepsilon}=0}.$$

- **Diagonal $a = I$: time evolution equations**

$$\left[\frac{d\theta}{d\tau} + i\,\frac{dp}{d\tau}\right] = \frac{i\kappa}{a^2}\frac{\partial\,\mathbf{H}[g^\varepsilon]}{\partial\varepsilon^I\left(e_I(v)\right)}\bigg|_{\vec{\varepsilon}=0}, \qquad \left[\frac{d\theta}{d\tau} - i\,\frac{dp}{d\tau}\right] = -\frac{i\kappa}{a^2}\frac{\partial\,\mathbf{H}[g_i^\varepsilon]}{\partial\varepsilon^I\left(e_I(v)\right)^*}\bigg|_{\vec{\varepsilon}=0}.$$

$\varepsilon^I\left(e_I(v)\right),\ \varepsilon^I\left(e_I(v)\right)^*$ **: longitudinal perturbations along the homogeneous and isotropic sector** $g^\varepsilon\left(e_I(v)\right) = e^{\left[\theta - ip + 2\varepsilon^I(e_I(v))\right]\tau^I/2}$

- **Off-diagonal $a \neq I$: constraint equations**

$$\frac{\partial\,\mathbf{H}[g^\varepsilon]}{\partial\varepsilon^a\left(e_I(v)\right)}\bigg|_{\vec{\varepsilon}=0} = \frac{\partial\,\mathbf{H}[g^\varepsilon]}{\partial\varepsilon^a\left(e_I(v)\right)^*}\bigg|_{\vec{\varepsilon}=0} = 0,$$

$\varepsilon^a\left(e_I(v)\right),\ \varepsilon^a\left(e_I(v)\right)^*$ **: transverse perturbations away from the homogeneous and isotropic sector** $g^\varepsilon\left(e_I(v)\right) = e^{[\theta - ip]\tau^I/2} e^{\varepsilon^a(e_I(v))\tau^a}$

Giesel-Thiemann's Hamiltonian

$$\hat{C}_v = \hat{C}_{0,v} + \frac{1+\beta^2}{2}\hat{C}_{L,v}, \quad \hat{K} = \frac{i}{\hbar\beta^2}\left[\sum_{v\in V(\gamma)}\hat{C}_{0,v}, \sum_{v\in V(\gamma)} V_v\right]$$

$$\hat{C}_{L,v} = \frac{16}{3\kappa\left(i\beta\ell_p^2/2\right)^3}\sum_{s_1,s_2,s_3=\pm 1} s_1 s_2 s_3 \varepsilon^{I_1 I_2 I_3}\,\mathrm{Tr}\left(\hat{h}\left(e_{v;I_1 s_1}\right)\left[\hat{h}\left(e_{v;I_1 s_1}\right)^{-1}, \hat{K}\right]\hat{h}\left(e_{v;I_2 s_2}\right)\left[\hat{h}\left(e_{v;I_2 s_2}\right)^{-1}, \hat{K}\right]\hat{h}\left(e_{v;I_3 s_3}\right)\left[\hat{h}\left(e_{v;I_3 s_3}\right)^{-1}, \hat{V}_v\right]\right)$$

For all Brown-Kuchar dust, Gaussian dust, gravity-scalar models,

$$\langle \tilde{\psi}^t_{g^\varepsilon} | \hat{\mathbf{H}} | \tilde{\psi}^t_{g^\varepsilon} \rangle = \mathbf{H}\left[g^\varepsilon\right] + O(\hbar)$$

Giesel and Thiemann 2007

Giesel-Thiemann volume $\hat{V}_v^{4q} = \langle \hat{Q}_v \rangle^{2q}\left[1 + \sum_{n=1}^{2k+1}(-1)^{n+1}\frac{q(1-q)\cdots(n-1+q)}{n!}\left(\frac{\hat{Q}_v^2}{\langle \hat{Q}_v\rangle^2} - 1\right)^n\right], \quad q > 0$

Insert in the classical discrete Hamiltonian, and compute linearized perturbations

$$\left[\frac{d\theta}{d\tau} + i\,\frac{dp}{d\tau}\right] = \frac{i\kappa}{a^2}\frac{\partial\,\mathbf{H}[g^\varepsilon]}{\partial\varepsilon^I\left(e_I(v)\right)}\bigg|_{\vec{\varepsilon}=0}, \qquad \left[\frac{d\theta}{d\tau} - i\,\frac{dp}{d\tau}\right] = -\frac{i\kappa}{a^2}\frac{\partial\,\mathbf{H}[g_i^\varepsilon]}{\partial\varepsilon^I\left(e_I(v)\right)*}\bigg|_{\vec{\varepsilon}=0}.$$

$$\frac{\partial\,\mathbf{H}[g^\varepsilon]}{\partial\varepsilon^a\left(e_I(v)\right)}\bigg|_{\vec{\varepsilon}=0} = \frac{\partial\,\mathbf{H}[g^\varepsilon]}{\partial\varepsilon^a\left(e_I(v)\right)*}\bigg|_{\vec{\varepsilon}=0} = 0,$$

The brute-force computation is carried out analytically in Mathematica.

The linear expansion uses the parallel computing environment of Mathematica with 30 parallel kernels on a CPU+GPU server.

The computation involves manipulation/cancellation of about 300k terms, and lasts about 2 hours.

Giesel-Thiemann's Hamiltonian

Results:

$$H(\theta,p)=\begin{cases}\frac{4a}{3\kappa\beta^2}\sqrt{2\beta p}\,\sin^2(\theta)\big[1-(1+\beta^2)\sin^2(\theta)\big], & \text{for Brown-Kuchař/Gaussian dusts}\\ \frac{\sqrt{2}\,a^2}{3\sqrt{\kappa}}p\sqrt{\sin^2(\theta)\big[1-(1+\beta^2)\sin^2(\theta)\big]}, & \text{for gravity-scalar}\,.\end{cases}$$

- **Diagonal $a = I$: time evolution equations**

$$\frac{d\theta}{d\tau}=-\frac{\kappa}{a^2}\frac{\partial H(\theta,p)}{\partial p},\quad \frac{dp}{d\tau}=\frac{\kappa}{a^2}\frac{\partial H(\theta,p)}{\partial\theta}$$

- **Off-diagonal $a \neq I$: constraint equations are satisfied automatically**

$$\frac{\partial\,\mathbf{H}[g^\varepsilon]}{\partial\varepsilon^a\left(e_I(v)\right)}\bigg|_{\vec{\varepsilon}=0}=\frac{\partial\,\mathbf{H}[g^\varepsilon]}{\partial\varepsilon^a\left(e_I(v)\right)^*}\bigg|_{\vec{\varepsilon}=0}=0,$$

Remarks:

- **involving cancellations between contributions from different vertices (using periodic boundary condition)**
- **Brown-Kuchař and Gaussian dust models give the same result since**

$$C_{j,v}[g]=0,\qquad C_{j,v}[g^\varepsilon]=O(\vec{\varepsilon}),\quad \text{and}\quad \mathbf{H}=\int_{\mathcal{S}}\mathrm{d}^3\sigma\sqrt{C(\sigma,\tau)^2-\frac{1}{4}\sum_{j=1}^{3}C_j(\sigma,\tau)C_j(\sigma,\tau)}$$

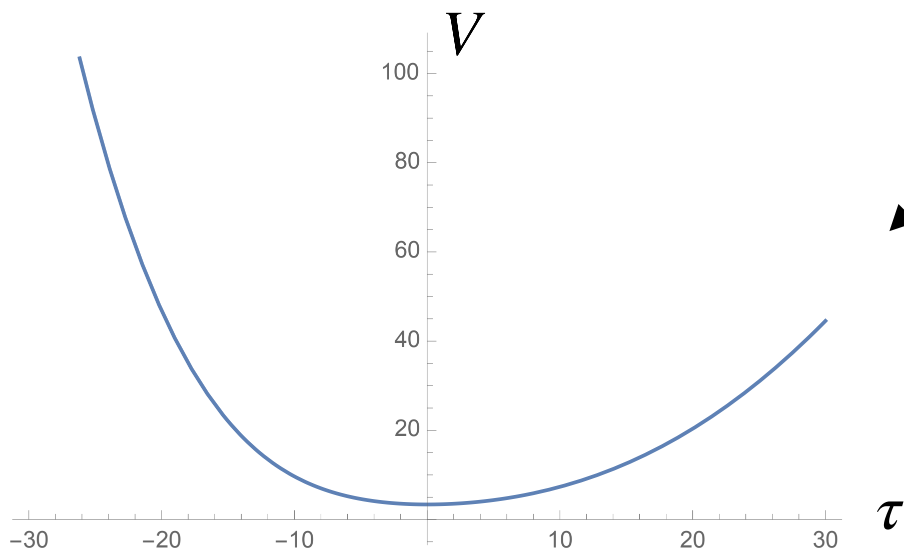
Giesel-Thiemann's Hamiltonian

Solution of evolution equations (Brown-Kuchar/Gaussian dust):

Change of variables $$p, \theta \quad \mapsto \quad V = \frac{a^3(\beta p)^{3/2}}{2\sqrt{2}}, \quad b = \theta$$

Conserved energy: $$\mathscr{E} = \frac{\mathbf{H}}{|V(\gamma)|}$$

Resolution of singularity and unsymmetric bounce



Reduces to FRW asymptotically

critical volume and density

$$V_c = \frac{27}{8}\beta^6(\beta^2+1)^3\kappa^3\mathscr{E}^3, \quad \rho_c = \mathscr{E}/V_c$$

(μ_0 – **scheme**)

Relate to the μ_0-scheme:
Change of variables

$$\theta = C\mu, \quad p = \frac{\mu^2}{a^2\beta}P$$

$$\frac{dP}{d\tau} = \frac{\partial}{\partial C}\mathbf{h}(C,P), \quad \frac{dC}{d\tau} = -\frac{\partial}{\partial P}\mathbf{h}(C,P),$$

$$\mathbf{h}(C,P) = \begin{cases} \frac{4}{3\mu^2\beta}\sqrt{2P}\ \sin^2(C\mu)\left[1-(1+\beta^2)\sin^2(C\mu)\right] & \text{for Brown-Kuchař/Gaussian dusts} \\ \frac{\sqrt{2\kappa}\,P}{3\mu}\sqrt{\sin^2(C\mu)\left[1-(1+\beta^2)\sin^2(C\mu)\right]} & \text{for gravity-scalar}. \end{cases}$$

same Hamiltonian as in Ding, Ma, and Yang 2009 and Dapor and Liegener 2017

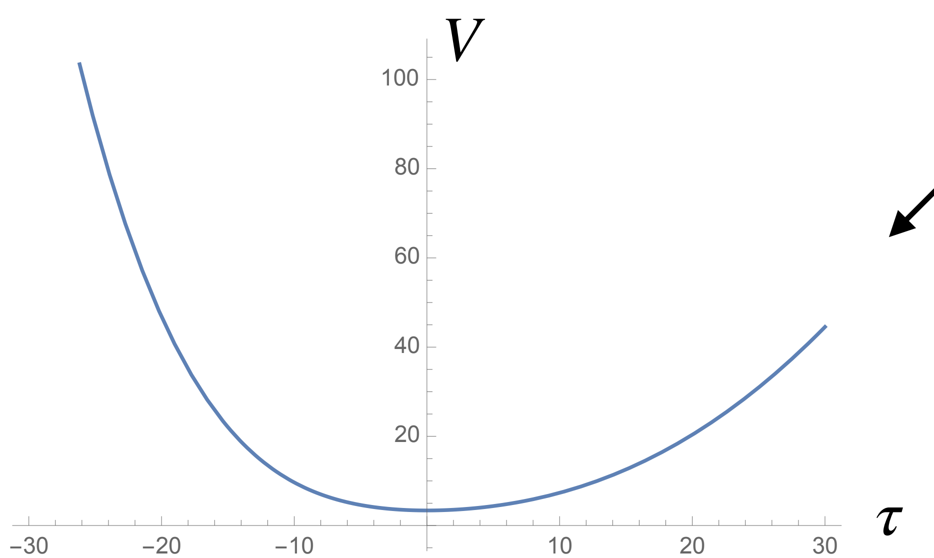
Giesel-Thiemann's Hamiltonian

Solution of evolution equations (Brown-Kuchar/Gaussian dust):

Change of variables $\quad p, \theta \quad \mapsto \quad V = \dfrac{a^3(\beta p)^{3/2}}{2\sqrt{2}}, \quad b = \theta$

Conserved energy: $\quad \mathscr{E} = \dfrac{\mathbf{H}}{|V(\gamma)|}$

Resolution of singularity and unsymmetric bounce



Reduces to FRW asymptotically

critical volume and density

$$V_c = \frac{27}{8}\beta^6(\beta^2+1)^3\kappa^3\mathscr{E}^3, \quad \rho_c = \mathscr{E}/V_c$$

(μ_0 – **scheme**)

A different change of variables: Liegener and Singh 2019

$$\theta = C\mu, \quad p = \frac{P}{a^2\beta}\frac{\sin^2(\mu C/2)}{C^2/4}$$

$$h\left(e_I(v)\right) = e^{\theta\tau_I/2}, \quad p^a\left(e_I(v)\right) = p\,\delta_I^a$$

But the resulting equation is different from Liegener and Singh 2019

Warsaw's Hamiltonian

Alesci, Assanioussi, Lewandowski, and Makinen 2015

$$\hat{C}_v = -\frac{1}{\beta^2}\hat{C}_{0,v} - \frac{1+\beta^2}{\kappa\beta^2}{}^3\hat{R}_v \qquad {}^3\hat{R}_v = \sum_{I\neq J}\sum_{s_1,s_2=\pm 1}\hat{L}_v(I,s_1;J,s_2)\left(\frac{2\pi}{\alpha} - \pi + \arccos\left[\frac{\hat{p}^a(e_{v;Is_1})\hat{p}^a(e_{v;Js_2})}{\hat{p}(e_{v;Is_1})\hat{p}(e_{v;Js_2})}\right]\right), \quad \alpha = 4$$

Alesci, Assanioussi, Lewandowski 2014

$$\hat{L}_v(I,s_1;J,s_2) = \frac{1}{\hat{V}_v}\sqrt{\epsilon^{abc}\hat{p}_b(e_{v;Is_1})\hat{p}_c(e_{v;Js_2})\epsilon^{ab'c'}\hat{p}_{b'}(e_{v;Is_1})\hat{p}_{c'}(e_{v;Js_2})}$$

Bianchi 2008

The semiclassical expansion in Giesel and Thiemann 2007 may not work for negative power of volume operator

$$\langle\tilde{\psi}^t_{g^\varepsilon}|\hat{\mathbf{H}}|\tilde{\psi}^t_{g^\varepsilon}\rangle = \mathbf{H}\left[g^\varepsilon\right] + O(\hbar)$$ **is a conjecture**

Assuming the conjecture is true

- **Diagonal $a = I$: time evolution equations**

$$\frac{d\theta}{d\tau} = -\frac{\kappa}{a^2}\frac{\partial H(\theta,p)}{\partial p}, \quad \frac{dp}{d\tau} = \frac{\kappa}{a^2}\frac{\partial H(\theta,p)}{\partial\theta} \qquad H[\theta,p] = \begin{cases} \frac{4a}{3\beta^2\kappa}\sqrt{2\beta p}\,\sin^2(\theta) & \textbf{for dust models} \\ \frac{\sqrt{2}\,a^2}{3\sqrt{\kappa}}p\sqrt{\sin^2(\theta)} & \textbf{for gravity-scalar} \end{cases}$$

- **Off-diagonal $a \neq I$: constraint equations are satisfied automatically**

$$\left.\frac{\partial\,\mathbf{H}[g^\varepsilon]}{\partial\varepsilon^a\left(e_I(v)\right)}\right|_{\vec{\varepsilon}=0} = \left.\frac{\partial\,\mathbf{H}[g^\varepsilon]}{\partial\varepsilon^a\left(e_I(v)\right)*}\right|_{\vec{\varepsilon}=0} = 0,$$

$${}^3R_v[g^\varepsilon] = O(\varepsilon^2)$$
at $\alpha = 4$

Reproduce standard LQC effective dynamics in μ_0-scheme: Symmetric bounce

Semiclassical Amplitude

$$A_{[g],[g']} := \left\langle \Psi^t_{[g]} \middle| \exp\left(-\frac{i}{\hbar} T \hat{\mathbf{H}}\right) \middle| \Psi^t_{[g']} \right\rangle_{\mathscr{H}_\gamma} = ||\psi^t_g|| \, ||\psi^t_{g'}|| \int dh \prod_{i=1}^{N+1} dg_i \, \nu[g] \; \exp\left(\frac{S[g,h]}{t}\right)$$

Given the initial/final condition (of phase space data), if the solution is unique,

$$A_{[g],[g']} \sim e^{S[g,h]/t} \Big|_{\textbf{solution}}$$

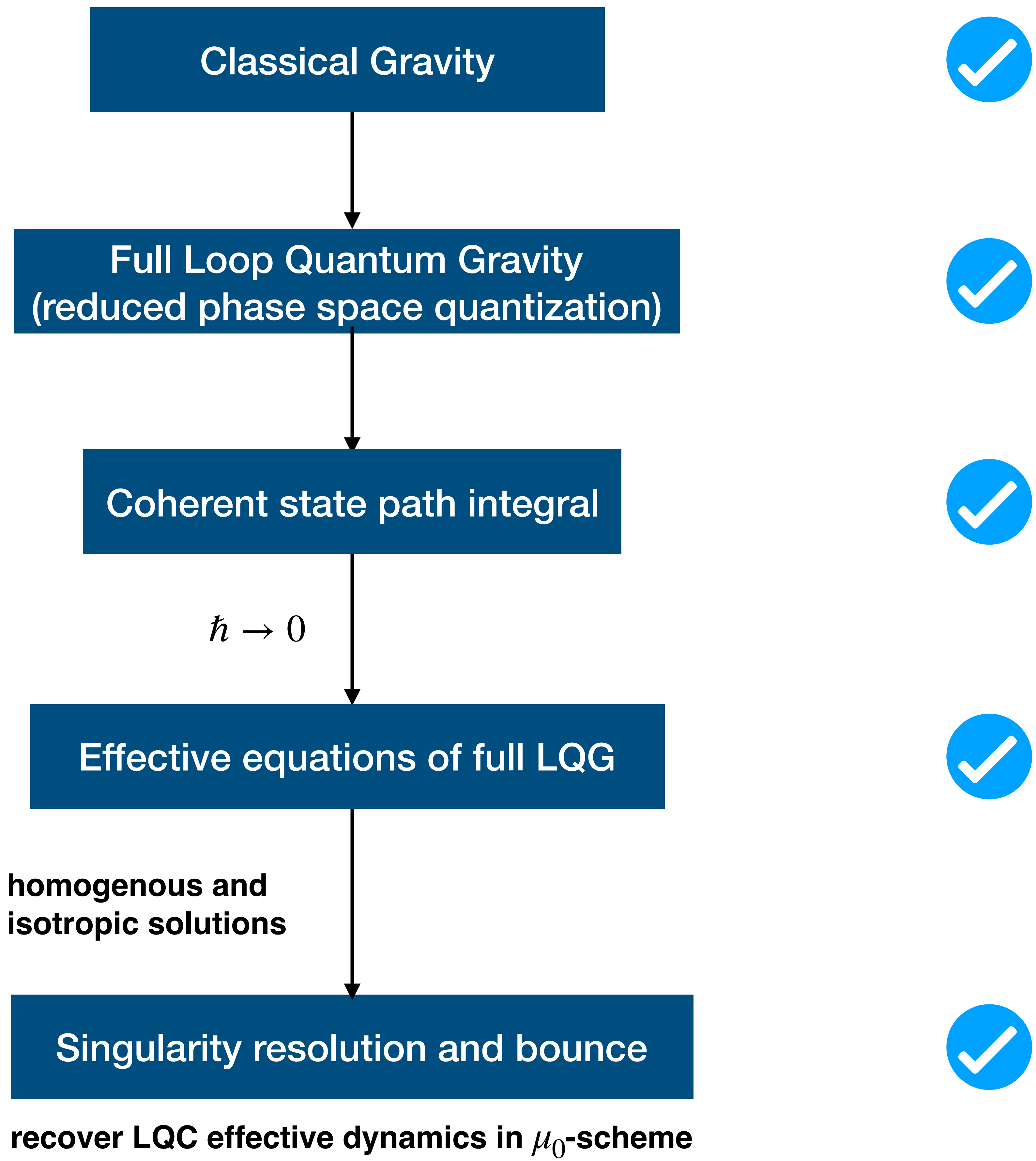
It is unclear if the solution is unique in the effective EOM of full LQG.

But the solution is indeed unique in the homogeneous and isotropic sector.

On-shell action:

$$S = -\, i \frac{\kappa |V(\gamma)|}{2a^2} \mathscr{E} T$$

This expression is independent of choices of Hamiltonian.


Classical Gravity
Full Loop Quantum Gravity
(reduced phase space quantization)
Coherent state path integral
$\hbar \to 0$
Effective equations of full LQG
homogenous and
isotropic solutions
Singularity resolution and bounce
recover LQC effective dynamics in μ_0-scheme

Outlook (I): $\bar{\mu}$-Scheme and Continuum Limit

Conserved energy: $\mathscr{E} = \dfrac{\mathbf{H}}{|V(\gamma)|}$

Critical volume and density depending on $\mathscr{E}$ $V_c = \dfrac{27}{8}\beta^6(\beta^2+1)^3\kappa^3\mathscr{E}^3, \quad \rho_c = \mathscr{E}/V_c \sim \mathscr{E}^{-2}$

Large $\mathscr{E}$ would imply large critical volume and small critical density. But is it necessarily true?

No, we have to take continuum limit $|V(\gamma)| \to \infty$, which sends $\mathscr{E} \to 0$, $V_c \to 0$ and $\rho_c \to \infty$.

$V_c \to 0$ indicates that the minimal area gap in LQG has to have an effect (hint of the $\bar{\mu}$-scheme)

In LQG, Minimal area gap is a non-perturbative quantum effect.

We may have to consider the quantum effective action of the path integral

$$e^{-\frac{i}{\hbar}W[J]} = \int D\phi\, e^{\frac{i}{\hbar}\left(S[\phi] + \int J\phi\right)}, \qquad \Gamma[\phi] = -W[J] - \int J\phi$$

and quantum effective equations $\delta\Gamma[\phi] = 0$

Alesci and Cianfrani 2013 obtains the $\bar{\mu}$-scheme from a random average of Hamiltonians over different lattices.

Outlook (I): Connection to Numerical Relativity

Effective EOMs from full LQG

$$\frac{z_{i+1,i}(e)\mathrm{Tr}\left[\tau^a g_{i+1}^\dagger(e) g_i(e)\right]}{\sqrt{x_{i+1,i}(e)-1}\sqrt{x_{i+1,i}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = \frac{i\kappa\Delta\tau}{a^2}\frac{\partial \mathbf{H}[g_i^\varepsilon]}{\partial \varepsilon_i^a(e)}\bigg|_{\varepsilon=0} \qquad i=1,\cdots,N$$

$$\frac{z_{i,i-1}(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_{i-1}(e)\right]}{\sqrt{x_{i,i-1}(e)-1}\sqrt{x_{i,i-1}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = -\frac{i\kappa\Delta\tau}{a^2}\frac{\partial \mathbf{H}[g^\varepsilon]}{\partial \varepsilon_i^a(e)^*}\bigg|_{\varepsilon=0}, \qquad i=2,\cdots,N+1$$

A set of dynamical evolution equation for full GR with natural discretization by LQG

- **ready for numerical simulation**
- **can be applied to all dynamical scenarios/spacetimes**

To do list:

- **gauge invariant cosmological perturbation theory: ongoing work**
  (comparing to Giesel, Hofmann, Thiemann, and Winkler 2007)
- **black holes and binaries**
- **gravitational waves**

……

Outlook (I): Connection to Numerical Relativity

Effective EOMs from full LQG

$$\frac{z_{i+1,i}(e)\mathrm{Tr}\left[\tau^a g_{i+1}^\dagger(e) g_i(e)\right]}{\sqrt{x_{i+1,i}(e)-1}\sqrt{x_{i+1,i}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = \frac{i\kappa\Delta\tau}{a^2}\frac{\partial \mathbf{H}[g_i^\varepsilon]}{\partial \varepsilon_i^a(e)}\bigg|_{\varepsilon=0} \qquad i=1,\cdots,N$$

$$\frac{z_{i,i-1}(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_{i-1}(e)\right]}{\sqrt{x_{i,i-1}(e)-1}\sqrt{x_{i,i-1}(e)+1}} - \frac{p_i(e)\mathrm{Tr}\left[\tau^a g_i^\dagger(e) g_i(e)\right]}{\sinh\left(p_i(e)\right)} = -\frac{i\kappa\Delta\tau}{a^2}\frac{\partial \mathbf{H}[g^\varepsilon]}{\partial \varepsilon_i^a(e)^*}\bigg|_{\varepsilon=0}, \qquad i=2,\cdots,N+1$$

A set of dynamical evolution equation for full GR with natural discretization by LQG

- **ready for numerical simulation**
- **can be applied to all dynamical scenarios/spacetimes**

To do list:

- **gauge invariant cosmological perturbation theory: ongoing work**
  (comparing to Giesel, Hofmann, Thiemann, and Winkler 2007)
- **black holes and binaries**
- **gravitational waves**

……

Thanks for your attention !