

# Quantum deformation of 4d spin foam models

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# Introduction

- Spin foam models : discretised functional integrals for TFTs of BF type and quantum gravity
- Models for theories with zero 'cosmological constant' are based on the representation theory of Lie groups
- Problem : generally these models diverge
- Natural regularisation : consider models based on the representation theory of quantum groups
- Our work : application of this procedure to the EPRL model

- 1 Three-dimensional models
  - Gravity in  $2 + 1$  dimensions
  - The Ponzano-Regge model
  - The Turaev-Viro model
  
- 2 Topological models in four dimensions
  - BF theory
  - The Ooguri model
  - The Crane-Yetter model
  
- 3 Quantum gravity models in four dimensions
  - Gravity as a constrained topological theory
  - The EPRL/FK models
  - Quantum deformation of the EPRL model

# Gravity in 2 + 1 dimensions

- First order gravity on a 3-dimensional manifold  $M$ :

- dual **co-frame**  $e : TM \rightarrow \mathbb{R}^3$
- $SO(\eta)$  **connection**  $A$  ( $\eta = (\pm, +, +)$  flat metric)

Let  $F_A = dA + \frac{1}{2}[A \wedge A]$  be the curvature of  $A$ , and  $\langle, \rangle$  the Killing form on  $\mathfrak{so}(\eta) \cong \mathbb{R}^3$

- Action for **2 + 1 gravity with cosmological constant  $\Lambda$**

$$S_\Lambda = \int_M \langle e \wedge F_A \rangle - \frac{\Lambda}{3} \langle e \wedge [e \wedge e] \rangle \quad (1)$$

- Goal : make sense of the formal quantity

$$\mathcal{Z}(M) = \int D[e]D[A] \exp iS_\Lambda[e, A] \quad (2)$$

- **Ponzano-Regge (PR) model** : **regularisation** of  $\mathcal{Z}(M)$  for Euclidean gravity with  $\Lambda = 0$

# Ponzano-Regge model [Ponzano, Regge '68; Freidel, Louapre '04; Barrett, Naish-Guzman '08]

- Let  $M$  be a closed, oriented triangulated 3-manifold
- PR model based on  $\text{Rep}(\text{SU}(2))$ . Data consists of :
  - Assignments  $I : e \mapsto I(e) \in \text{Irrep}(\text{SU}(2))$  to the edges  $e$  of  $M$
  - A state space for each triangle  $\Delta$  of  $M$

$$H_{\Delta} = \text{Hom}\left(\bigotimes_{e \in \partial\Delta} I(e), \mathbb{C}\right) \ni \alpha,$$

- An amplitude  $A_t : \bigotimes_{\Delta \in \partial t} H_{\Delta} \rightarrow \mathbb{C}$  for each tetrahedron  $t$

$$A_t(\alpha \otimes \beta \otimes \gamma \otimes \delta) = \{6j\}_t$$

- PR model : 
$$PR(M) = \sum_I \prod_e \dim I(e) \prod_t \{6j\}_t \quad (3)$$
- Problem : the above infinite sum generally diverges

[Bonzom, Smerlak '10]

# The Turaev-Viro model [Turaev, Viro '92]

- Goal : regularise the divergencies of the PR model
- Idea : the quantum group  $U_q(\mathfrak{su}(2))$  with  $q^r = 1$  admits only a **finite number** of **irreducible representations** [Arnaudon, Roche '89]
  - Natural regularisation of the PR model : replace  $SU(2)$  by its quantum deformation  $U_q(\mathfrak{su}(2))$  at root of unity
- Irreducible representations of  $U_q(\mathfrak{su}(2))$  (after purification):

$$l \in \left\{ 0, \frac{1}{2}, 1, \dots, \frac{r-2}{2} \right\}, \quad q = \exp 2i\pi/r$$

- Constructing a model based on  $\text{Rep}(U_q(\mathfrak{su}(2)))$  leads to a **finite** model: the Turaev-Viro invariant

$$TV_q(M) = K \sum_l \prod_e [\dim l(e)]_q \prod_t \{6j_q\}_t \quad (4)$$

## Relation to the cosmological constant $\Lambda$

- Using the equivalence [Witten '88] between **2 + 1 gravity** and **Chern-Simons theory** one can show that :

$$TV_q(M) \propto \int D[e]D[A] \exp iS_\Lambda[e, A], \quad \text{if } q = \exp il_p/l_c \quad (5)$$

- Asymptotics** of the quantum  $6j$  symbol when  $l(e) \rightarrow \infty$  :

[Mizoguchi, Tada '92]

$$\{6j_q\}_t \sim \frac{1}{\sqrt{12\pi V_t}} \cos\left(S_t + \frac{\pi}{4}\right), \quad V_t = \text{volume}(t), \quad (6)$$

where  $S_t$  is the **Regge action with cosmological constant  $\Lambda$**  for the tetrahedron  $t$

$$S_t = \sum_{e \in \partial t} \theta_e l(e) - \frac{1}{l_c^2} V_t \quad (7)$$

## 4d BF theory [Horowitz '89; Cattaneo, Cotta-Ramusino, Fröhlich, Martellini '95; Baez '95]

- BF theory with semi-simple Lie group  $G$  on a 4-manifold  $M$ :
  - **B field**  $B \in \Omega^2(M) \otimes \mathfrak{g}$
  - **connection**  $A$  on a principal  $G$ -bundle over  $M$

Let  $F_A = dA + \frac{1}{2}[A \wedge A]$  be the curvature of  $A$ , and  $\langle, \rangle$  the Killing form on  $\mathfrak{g}$

- Action of 4d **BF theory with cosmological constant**  $\Lambda$

$$S_\Lambda = \int_M \langle B \wedge F_A \rangle - \frac{\Lambda}{12} \langle B \wedge B \rangle \quad (8)$$

- Goal : make sense of the formal quantity

$$\mathcal{Z}(M) = \int D[B]D[A] \exp iS_\Lambda[B, A] \quad (9)$$

# The Ooguri model [Ooguri '92]

- **Ooguri model** : regularisation of  $\mathcal{Z}(M)$  in the case of  $G = \text{SU}(2)$  and  $\Lambda = 0$
- Model based on  $\text{Rep}(\text{SU}(2))$ 
  - The **assignments** are now to the **triangles**  $\Delta$  and **tetrahedra**  $t$  of a triangulation of  $M$

$$I : \Delta \mapsto I(\Delta) \in \text{Irrep}(\text{SU}(2)), \quad t \mapsto \alpha_t \in H_t = \text{Hom}\left(\bigotimes_{\Delta \in \partial t} I(\Delta), \mathbb{C}\right)$$

- The **amplitude** for the 4-simplexes  $\sigma$  are given by **15j-symbols**
- Partition function :

$$O(M) = \sum_{I, J} \prod_{\Delta} \dim I(\Delta) \prod_t \dim J(t)^{-1} \prod_{\sigma} \{15j\}_{\sigma} \quad (10)$$

# The Crane-Yetter model [Crane, Yetter, Kauffman '93]

- The Ooguri model **diverges** in general
- It can be regularised by considering a model based on  $\text{Rep}(U_q(\mathfrak{su}(2)))$  with  $q^r = 1$

$$CY_q(M) = K \sum_{I,J} \prod_{\Delta} [\dim I(\Delta)]_q \prod_t [\dim J(t)]_q^{-1} \prod_{\sigma} \{15j_q\}_{\sigma} \quad (11)$$

- The Crane-Yetter model is **finite** and provides an **invariant of topological 4-manifolds**
- It is related to  $SU(2)$  **BF theory** on  $M$  with **cosmological constant**  $\Lambda$  via Chern-Simons theory on  $\partial M$  [Roberts '93; Baez '95]

$$CY_q(M) \propto \int D[B]D[A] \exp iS_{\Lambda}[B, A], \quad \text{if } q = \exp i\ell_p^2/\ell_c^2 \quad (12)$$

# Gravity as a constrained BF theory [Plebanski '77; Freidel, De Pietri '98]

- First order gravity on a 4-dimensional manifold  $M$ :
  - dual **co-frame**  $e : TM \rightarrow \mathbb{R}^4$
  - $SO(\eta)$  **connection**  $A$  ( $\eta = (\epsilon, +, +, +)$ ,  $\epsilon = \pm 1$ , flat metric)
- Action for **3 + 1 gravity with cosmological constant**  $\Lambda$

$$S_\Lambda = \int_M \langle *(e \wedge e) \wedge F_A \rangle - \frac{\Lambda}{12} \langle *(e \wedge e) \wedge e \wedge e \rangle \quad (13)$$

- Suggests that GR could be cast as a BF theory :

$$S_\Lambda = \int_M \langle B[e] \wedge F_A \rangle - \frac{\epsilon \Lambda}{12} \langle B[e] \wedge *B[e] \rangle, \quad \text{with } B[e] = *(e \wedge e)$$

- **Gravity**  $\equiv$  **BF** theory + **constraints** on the  $B$  field

$$S_\Lambda^{\text{Plebanski}} = \int_M \langle B \wedge F_A \rangle - \frac{\epsilon \Lambda}{12} \langle B \wedge *B \rangle + \mathcal{C}(B) \quad (14)$$

# The EPRL/FK models [Engle, Pereira, Rovelli, Livine '08; Freidel, Krasnov '08]

- **EPRL/FK models** : spin foam models for **Plebanski's theory** with  $\Lambda = 0$  and Immirzi parameter  $\gamma$
- Idea : implement the Plebanski constraints as a choice of measure in the path integral for BF theory
- Concretely :
  - Consider a generalised **Ooguri model** for  $\text{Rep}(\text{SO}(\eta))$
  - Impose **constraints on the data of the model**, i.e., at the level of **representation labels**  $I(\Delta)$  and **state spaces**  $H_t$
- Questions :
  - Are the EPRL/FK models finite ? [Perini, Rovelli, Speziale '09]
  - How to introduce a **cosmological constant** ?  
[Bianchi, Krajewski, Rovelli, Vidotto '11]
- Possible answer : consider the **quantum deformation of the models** [Smolin '95; Major, Smolin '96; Smolin '02; Rovelli '10]

# Quantum deformation of the EPRL model

- Our work : construction and analysis of a  $q$ -deformation of the **Euclidean and Lorentzian** versions of the **EPRL model**

- The Euclidean model is based on

$$U_q(\mathfrak{spin}(4)) = U_q(\mathfrak{su}(2)) \otimes U_q(\mathfrak{su}(2)), \quad \text{with } q \text{ root of unity}$$

- The Lorentzian model on

$$U_q(\mathfrak{sl}(2, \mathbb{C})) = \mathcal{D}U_q(\mathfrak{su}(2)) = U_q(\mathfrak{su}(2)) \hat{\otimes} F_q(SU(2))^{op}, \quad \text{with } q \text{ real}$$

- **Basic ingredients** for the construction of the model :
  - 1 A given **subset of representations**  $\mathcal{I} \subset \text{Rep}(U_q(\mathfrak{so}(\eta)))$   
( $q$ -analogue of EPRL representations)
  - 2 A **specific class of intertwiners**  $\iota$  between the elements of  $\mathcal{I}$   
( $q$ -analogue of EPRL intertwiners)
- From here on, we will focus on the Lorentzian model

## $q$ -Lorentzian EPRL representations

- **Unitary, irreducible representations** (of the principal series) of  $U_q(\mathfrak{sl}(2, \mathbb{C}))$ ,  $q = e^{-\kappa}$  [Pusz '93, Buffenoir, Roche '99] :

$$\text{Irrep}(U_q(\mathfrak{sl}(2, \mathbb{C}))) = \{(n, p), n \in \mathbb{Z}/2, p \in [0, 4\pi/\kappa[ \} \quad (15)$$

- Let  $\gamma \in \mathbb{R}$  be a parameter. An **EPRL representation** is a **map**

$$\phi_\gamma : \text{Irrep}(U_q(\mathfrak{su}(2))) \rightarrow \text{Irrep}(U_q(\mathfrak{sl}(2, \mathbb{C}))),$$

defined by

$$K \mapsto (n(K), p(K)) = (K, \gamma K) \quad (16)$$

- Remark: the **pre-image** of the map  $\phi_\gamma$  is **restricted** to

$$\mathcal{L} = \{K \in \mathbb{N}/2 \mid K < 4\pi/\gamma\kappa\} \subset \text{Irrep}(U_q(\mathfrak{su}(2)))$$

## $q$ -Lorentzian EPRL interwiner : Definition I.

- Notation : principal representations  $(\pi_\alpha, V_\alpha)$ ,  $\alpha = (n, p)$
- The EPRL representation  $\alpha(K)$  factorises as

$$V_{\alpha(K)} = \bigoplus_{I=K}^{\infty} V_I, \quad V_I : U_q(\mathfrak{su}(2))\text{-module} \quad (17)$$

- Let  $f_\alpha^K : V_\alpha \rightarrow V_K$  be the projection on the lowest weight factor. The dual map induces an embedding

$$f^* : \text{Hom}_{U_q(\mathfrak{su}(2))}(\bigotimes_{i=1}^n V_{K_i}, \mathbb{C}) \rightarrow \text{Hom}_{U_q(\mathfrak{sl}(2, \mathbb{C}))}(\bigotimes_{i=1}^n V_{\alpha_i}, \mathbb{C})$$

- To all  $U_q(\mathfrak{su}(2))$ -intertwiner  $\Lambda_K : \bigotimes_{i=1}^n V_{K_i} \rightarrow \mathbb{C}$ , this map associates a quantum EPRL intertwiner  $\iota_\alpha(K)$

$$\iota_\alpha(K) = f^*(\Lambda_K) = \Lambda_K \circ \bigotimes_{i=1}^n f_{\alpha_i}^{K_i} \circ T_{\alpha_1, \dots, \alpha_n} \quad (18)$$

## $q$ -Lorentzian EPRL interwiner : Definition II.

- $T_{\alpha_1, \dots, \alpha_n}$  is the  $q$ -analogue of the classical expression

$$\int_{\mathrm{SL}(2, \mathbb{C})} dX \left( \bigotimes_{i=1}^n \pi_{\alpha_i} \right) (X) \quad (19)$$

- Let  $h : F_q(\mathrm{SL}(2, \mathbb{C})) \rightarrow \mathbb{C}$  be a Haar measure on the Hopf algebra  $F_q(\mathrm{SL}(2, \mathbb{C}))$  dual to  $U_q(\mathfrak{sl}(2, \mathbb{C}))$  [Buffenoir, Roche '99]

$$\text{Ex. : } f \in F(\mathrm{SL}(2, \mathbb{C})), \quad h(f) = \int_{\mathrm{SL}(2, \mathbb{C})} dX f(X)$$

- $T_{\alpha_1, \dots, \alpha_n}$  is defined as [Noui, Roche '02]

$$T_{\alpha_1, \dots, \alpha_n} = \sum_A \left( \bigotimes_{i=1}^n \pi_{\alpha_i} \right) (\Delta^{(n)}(x^A)) h(x_A), \quad (20)$$

where  $\{x^A\}_A$  is a basis of  $U_q(\mathfrak{sl}(2, \mathbb{C}))$  and  $\{x_A\}_A$  is the dual basis of  $F_q(\mathrm{SL}(2, \mathbb{C}))$

## $q$ -Lorentzian EPRL intertwiner : Properties

- Theorem.** 1. Let  $\{e_a^I(\alpha) \mid I \in \mathbb{N}, I \geq |n|, a = -I, \dots, I\}$  be a basis of  $V_\alpha$ . The evaluation of the 4-valent *quantum EPRL intertwiner*  $\iota_\alpha(\bigotimes_{i=1}^4 e_{a_i}^{I_i}(\alpha_i))$  is a multiple series which *converges absolutely*
  - The linear map  $\iota_\alpha : \bigotimes_{i=1}^4 V_{\alpha_i} \rightarrow \mathbb{C}$  is a *intertwiner* for the quantum Lorentz group
- Proposition.** Let  $R$  be the matrix of the quantum double  $DU_q(\mathfrak{su}(2))$  and  $c_{\alpha_2, \alpha_1} : V_{\alpha_2} \otimes V_{\alpha_1} \rightarrow V_{\alpha_1} \otimes V_{\alpha_2}$  with

$$c_{\alpha_2, \alpha_1} = \tau_{\alpha_2, \alpha_1} \circ (\pi_{\alpha_2} \otimes \pi_{\alpha_1})(R), \quad (21)$$

the associated braiding. The *quantum EPRL intertwiner*  $\iota_\alpha$  is *not invariant* under the *action of  $c$*  :

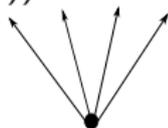
$$\iota_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \circ c_{\alpha_2, \alpha_1} \neq \iota_{\alpha_2 \alpha_1 \alpha_3 \alpha_4} \quad (22)$$

## Graphical calculus : ingredients

- 4-simplex amplitude defined using graphical calculus
- Elements for the diagrammatic calculus :

- **EPRL tensors** : elements of  $\left[ \bigotimes_{i=1}^4 V_{\alpha_i} \right] \otimes F_q(\mathrm{SL}(2, \mathbb{C}))$

$$\psi^\alpha = \left[ \sum_A \left( \bigotimes_{i=1}^4 \pi_{\alpha_i(K_i)} \left( \Delta^{(4)}(x^A) \right) \right) \circ \bigotimes_{i=1}^4 f_{K_i}^{\alpha_i} \circ \Lambda^K \right] \otimes x_A :=$$



where  $\Lambda^K \in \mathrm{Hom}_{U_q(\mathfrak{su}(2))} \left( \mathbb{C}, \bigotimes_{i=1}^4 V_{K_i} \right)$  and  $f_K^\alpha : V_K \rightarrow V_\alpha$

- An invariant **bilinear form**  $\beta : V_\alpha \times V_\alpha \rightarrow \mathbb{C}$

$$\beta :=$$

- The **braiding**  $c_{\alpha, \beta} : V_\alpha \otimes V_\beta \rightarrow V_\beta \otimes V_\alpha$

$$c_{\alpha, \beta} :=$$



## The model

- Let  $M$  be a closed, oriented triangulated 4-manifold
- Model based on  $\text{Rep}(U_q(\mathfrak{sl}(2, \mathbb{C})))$ . Data consists of :
  - An assignment  $\alpha : \Delta \mapsto \alpha(\Delta) \in \text{Irrep}(U_q(\mathfrak{sl}(2, \mathbb{C})))$  of an EPRL representation to each triangle  $\Delta$  of  $M$
  - A state space of EPRL tensors for each tetrahedron  $t$  of  $M$

$$H_t = \left( \bigotimes_{\Delta \in \partial t} \alpha(\Delta) \right) \otimes F_q(\text{SL}(2, \mathbb{C}))$$

- An amplitude  $A_\sigma : \bigotimes_{t \in \partial \sigma} H_t \rightarrow \mathbb{C}$  for each 4-simplex  $\sigma$ 

$$A_\sigma(\Psi_1 \otimes \dots \otimes \Psi_5) = \text{ev}(\Gamma_5) \in \mathbb{C}$$
- The resulting model is *finite*. It is given by

$$\mathcal{Z}_q(M) = \sum_{K \in \mathcal{L}} \sum_J \prod_{\Delta} [\dim K(\Delta)]_q \prod_{\sigma} A_\sigma(\{K(\Delta)\}, \{J(t)\}) \quad (25)$$

## Conclusion

- We have **constructed and analysed** a **q-deformation** of the **EPRL model**. We have :
  - Generalised the classical constructions to models based on  $U_q(\mathfrak{su}(2)) \otimes U_q(\mathfrak{su}(2))$  and  $\mathcal{D}U_q(\mathfrak{su}(2))$
  - Defined the EPRL intertwiner, studied its convergence and properties under braiding
  - Constructed a convergent amplitude for the 4-simplexes
- Open question: **relation to the cosmological constant  $\Lambda$  ?**
  - If  $q = e^{-l_p^2/l_c^2}$  (Lorentzian), bound on the area of the triangles

$$A(\Delta) < 32\pi^2 l_c^2 \quad (l_p \ll l_c) \quad (26)$$

- Need for an asymptotic formula [Ding, Han '11]