Relation Between Schrödinger and Polymer Quantum Mechanics

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MOTIVATION

- **What is the relation between the polymer and Schrödinger representations?**
- **Can we consider a continuum limit?**
- **What is then the continuum limit?**
- **What if it exists?**
- **What if there isn't any?**
- **SHO vs LQC, are they similar?**

PLAN OF THE TALK

1. Motivation
2. From Schrödinger to Polymer
3. Polymer Representation
4. Continuum Limit
5. Simple Harmonic Oscillator
6. Free Particle
7. Polymer Quantum Cosmology

2. Preliminaries

From loop quantum gravity, we have learned that the only way to construct diffeomorphism invariant theories is to start with exponentiated objects, like holonomies: $h_e(A) = \mathcal{P} \exp(\oint A)$.

Is there a price? Yes! The quantum theory becomes discontinuous.

This means that for a system with $\hat{p}$ and $\hat{q}$ as fundamental coordinates, one of them becomes ill-defined.

But if the Hamiltonian is of the form $H = p^2 + V(q)$, we can not define it on the Kinematical Hilbert space $\mathcal{H}$. Can we do something about it?
Not directly!

But we can attempt the next best thing:

Regularize the operator!

But, how? General strategy: approximate the non-existing operator by a different (finite) operator that does exist.

In general, there will be a regulator that does not go away. If we don't have diffeo invariance (as in LQG *a la* Thiemann), how can we get rid of the regulator?

The objective of this talk is to explore this issue. Namely, can we take this regulator away and thus arrive at the 'continuum limit'?

Is there a generic result? Case by case?

From Schrödinger to Polymer

Our starting point will be $\Gamma = \mathbb{R}^2$ with coordinates (q, p) thereon.
Here the starting point is the algebra generated by the exponentiated versions of $\hat{q}$ and $\hat{p}$ that are denoted by,

$$U(\alpha) = e^{i(\alpha\,\hat{q})/\hbar} \qquad ; \qquad V(\beta) = e^{i(\beta\,\hat{p})/\hbar}$$

The CCR now become

$$U(\alpha) \cdot V(\beta) = e^{(-i\alpha\,\beta)/\hbar} V(\beta) \cdot U(\alpha) \tag{1}$$

and the rest of the product is,

$$U(\alpha_1) \cdot U(\alpha_2) = U(\alpha_1 + \alpha_2) \quad ; \quad V(\beta_1) \cdot V(\beta_2) = V(\beta_1 + \beta_2)$$

The Weyl algebra $\mathcal{W}$ is generated by taking finite linear combinations of the generators $U(\alpha_i)$ and $V(\beta_i)$. Quantization means finding an unitary representation of the Weyl algebra $\mathcal{W}$ on a Hilbert space $\mathcal{H}'$.

A 'Fock type' representation involves a complex structure J, a linear mapping from Γ to itself such that $J^2 = -1$. In 2 dimensions, all the freedom is in the

choice of a parameter d with dimensions of length. For $k = p/\hbar$, we have

$$J_d : (q, k) \mapsto (-d^2\, k, q/d^2)$$

This object together with the symplectic structure: $\Omega((q,p);(q',p')) = q\,p' - p\,q'$ define an inner product on Γ: $g_d(\cdot\,;\,\cdot) = \Omega(\cdot\,;J_d\,\cdot)$ such that:

$$g_d((q,p);(q',p')) = \frac{1}{d^2}\,q\,q' + \frac{d^2}{\hbar^2}\,p\,p'$$

The representation of the Weyl algebra is then, when acting of functions $\phi(q)$:

$$\hat{U}(\alpha)\cdot\phi(q) := (e^{i\alpha\,q/\hbar}\,\phi)(q)$$

and,

$$\hat{V}(\beta)\cdot\phi(q) := e^{\frac{\beta}{d^2}(q-\beta/2)}\,\phi(q-\beta)$$

One defines an algebraic state (a positive linear functional) $\omega_d : \mathcal{W} \to \mathbb{C}$, that must coincide with the vacuum expectation value in the Hilbert space: $\omega_d(a) = \langle\hat{a}\rangle_0$, for all $a \in \mathcal{W}$. In our case this the specification of J induces

such a unique state ω_d that yields,

$$\langle \hat{U}(\alpha) \rangle_{\rm vac} = e^{-\frac{1}{4}\frac{d^2\alpha^2}{\hbar^2}} \qquad and \qquad \langle \hat{V}(\beta) \rangle_{\rm vac} = e^{-\frac{1}{4}\frac{\beta^2}{d^2}}$$

Wave functions belong to the space $L^2(\mathbb{R}, {\rm d}\mu_d)$, where the measure that dictates the inner product in this representation is given by,

$$ {\rm d}\mu_d = \frac{1}{d\sqrt{\pi}}\, e^{-\frac{q^2}{d^2}}\, {\rm d}q$$

In this representation, the vacuum is given by the identity function $\phi_0(q) = 1$ that is, just as any other plane wave, normalized.

There is an isometric isomorphism K that maps the d-representation in $\mathcal{H}_d$ to the standard Schrödinger representation in $\mathcal{H}_{\rm schr}$ by:

$$\psi(q) = K \cdot \phi(q) = \frac{e^{-\frac{q^2}{2\,d^2}}}{d^{1/2}\pi^{1/4}}\, \phi(q) \;\in\; \mathcal{H}_{\rm schr} = L^2(\mathbb{R}, {\rm d}q)$$

All d-representations are unitarily equivalent. Note also that the vacuum now

becomes

$$\psi_0(q) = \frac{1}{d^{1/2}\pi^{1/4}}\, e^{-\frac{q^2}{2\,d^2}}\,,$$

The $1/d \mapsto 0$ limit.

The measure behaves as:

$$\mathrm{d}\mu_d = \frac{1}{d\sqrt{\pi}}\, e^{-\frac{q^2}{d^2}}\, \mathrm{d}q \mapsto \frac{1}{d\sqrt{\pi}}\, \mathrm{d}q$$

The inner product between two such states is given by

$$\langle \phi_\alpha, \phi_\lambda \rangle_d = \int \mathrm{d}\mu_d\, e^{-\frac{i\alpha q}{\hbar}}\, e^{\frac{i\lambda q}{\hbar}} = e^{-\frac{(\lambda-\alpha)^2\, d^2}{4\,\hbar^2}}$$

It is immediate to see that in the $1/d \mapsto 0$ limit the inner product becomes,

$$\langle \phi_\alpha, \phi_\lambda \rangle_d \mapsto \delta_{\alpha,\lambda} \tag{2}$$

with $\delta_{\alpha,\lambda}$ being Kronecker's delta. We see then that the plane waves $\phi_\alpha(q)$ become an orthonormal basis for the new Hilbert space.
Let us see the action of the operator $\hat{V}(\beta)$ on the basis $\phi_\alpha(q)$:

$$\hat{V}(\beta)\cdot\phi_\alpha(q) = e^{-\frac{\beta^2}{2d^2}-i\frac{\alpha\beta}{\hbar}}\; e^{(\beta/d^2+i\alpha/\hbar)q}$$

which in the limit $1/d \mapsto 0$ goes to,

$$\hat{V}(\beta)\cdot\phi_\alpha(q) \mapsto e^{i\frac{\alpha\beta}{\hbar}}\,\phi_\alpha(q)$$

that is continuous on β. Thus, in the limit, the operator $\hat{p} = -i\hbar\partial_q$ is well defined. Also, note that in this limit the operator $\hat{p}$ has $\phi_\alpha(q)$ as its eigenstate with eigenvalue given by α:

$$\hat{p}\cdot\phi_\alpha(q) \mapsto \alpha\,\phi_\alpha(q)$$

Polymer Quantum Mechanics: Kinematics

Define abstract kets $|\mu\rangle$ labeled by a real number μ in the Hilbert space $\mathcal{H}_{\rm poly}$. A generic 'cylinder states' corresponds to a choice of a finite collection of numbers $\nu_i \in \mathbb{R}$ with $i = 1, 2, \ldots, N$. Associated to this choice, there are N vectors $|\mu_i\rangle$, so we can take a linear combination of them

$$|\psi\rangle = \sum_{i=1}^{N} a_i \, |\mu_i\rangle \tag{3}$$

The polymer inner product between the kets is given by,

$$\langle \nu | \mu \rangle = \delta_{\nu,\mu} \tag{4}$$

The kets are orthogonal to each other (when $\nu \neq \mu$) and they are normalized ($\langle \mu | \mu \rangle = 1$). Immediately, this implies that, given any two vectors $|\phi\rangle = \sum_{j=1}^{M} b_j |\nu_j\rangle$ and $|\psi\rangle = \sum_{i=1}^{N} a_i \, |\mu_i\rangle$, the inner product between them is given by,

$$\langle \phi | \psi \rangle = \sum_i \sum_j \bar{b}_j \, a_i \, \langle \nu_j | \mu_i \rangle = \sum_k \bar{b}_k \, a_k$$

where the sum is over k that labels the intersection points between the set of labels $\{\nu_j\}$ and $\{\mu_i\}$.

The Hilbert space $\mathcal{H}_{\rm poly}$ is the Cauchy completion of finite linear combination of the form (3) with respect to the inner product (4). $\mathcal{H}_{\rm poly}$ is non-separable. There are two basic operators on this Hilbert space: the 'label operator' $\hat{\varepsilon}$:

$$\hat{\varepsilon} \, |\mu\rangle := \mu \, |\mu\rangle$$

and the displacement operator $\hat{\mathbf{s}}\,(\lambda)$,

$$\hat{\mathbf{s}}\,(\lambda) \, |\mu\rangle := |\mu - \lambda\rangle$$

The operator $\hat{\varepsilon}$ is symmetric and the operator(s) $\hat{\mathbf{s}}(\lambda)$ unitary but discontinuous.

Polymer Quantum Mechanics: Dynamics

Consider the case of a particle of mass m in a potential $V(q)$:

$$H = \frac{1}{2m}\, p^2 + V(q)$$

We are in trouble. Why? In the polymer representation we can either represent q or p, but not both! What to do then?
The only thing possible: approximate the non-existing term by a well defined function that *can* be quantized and hope for the best.
Let us chose the position to be discrete, so $\hat{p}$ does not exist. With this choice, the kinetic term $\hat{p}^2/2m$ has to be approximated, for any potential.
How is this done? The standard prescription is to define, on the configuration space $\mathcal{C}$, a regular 'graph' γ_{μ_0}. This consists of a numerable set of points, equidistant, and characterized by a parameter μ_0 that is the (constant) separation between points. The simplest example would be to consider the set $\gamma_{\mu_0} = \{q \in \mathbb{R} \mid q = n\,\mu_0\ , \forall\ n \in \mathbb{Z}\}$.

The basic kets are: $|\mu_n\rangle$ that correspond to labels μ_n belonging to the graph γ_{μ_0}, that is, $\mu_n = n\,\mu_0$.
Thus, we shall only consider states of the form,

$$|\psi\rangle = \sum_n a_n\,|\mu_n\rangle\,. \tag{5}$$

This 'small' Hilbert space $\mathcal{H}_{\gamma_{\mu_0}}$, the graph Hilbert space, belongs to the 'large' polymer Hilbert space $\mathcal{H}_{\rm poly}$ but is separable. The condition for a state of the form (5) to belong to the Hilbert space $\mathcal{H}_{\gamma_{\mu_0}}$ is that: $\sum_n |a_n|^2 < \infty$.
What about $\hat{p}^2/2m$? If we want to remain in the graph γ, and not create 'new points', then one is constrained to considering operators that displace the kets by just the right amount. That is we want the basic shift operator $\hat{\mathbf{s}}(\lambda)$ to be such that it maps the ket with label $|\mu_n\rangle$ to the next ket, namely $|\mu_{n+1}\rangle$. Fix, once and for all, the value of the allowed parameter λ to be $\lambda = \mu_0$.

$$\hat{\mathbf{s}}(-\mu_0)\cdot|\mu_n\rangle = |\mu_n + \mu_0\rangle = |\mu_{n+1}\rangle$$

This basic 'shift operator' is the building block for approximating any (polyno-

mial) function of p. The function p is approximated by,

$$p \approx \frac{\hbar}{\mu_0} \sin\left(\frac{\mu_0\, p}{\hbar}\right) = \frac{\hbar}{2i\mu_0}[\mathbf{s}(\mu_0) - \mathbf{s}(-\mu_0)]$$

where the approximation is good for $p << \hbar/\mu_0$. Thus, the regulated operator $\hat{p}_{\mu_0}$ that depends on the 'regulator' μ_0 is:

$$\hat{p}_{\mu_0} \cdot |\mu_n\rangle := \frac{i\hbar}{2\mu_0}\left(|\mu_{n+1}\rangle - |\mu_{n-1}\rangle\right) \tag{6}$$

To regulate the operator $\hat{p}^2$, there is an operator that only involves shifting once again:

$$\hat{p}^2_{\mu_0} \cdot |\mu_n\rangle := \frac{\hbar^2}{\mu_0^2}\left(2|\mu_n\rangle - |\mu_{n+1}\rangle - |\mu_{n-1}\rangle\right) \tag{7}$$

which corresponds to the approximation $p^2 \approx \frac{2\hbar^2}{\mu_0^2}(1 - \cos(\mu_0\, p/\hbar))$, valid also in the regime $p << \hbar/\mu_0$. The operator $\hat{H}_{\mu_0}$, that 'lives' on $\mathcal{H}_{\gamma_{\mu_0}}$ is,

$$\hat{H}_{\mu_0} := \frac{1}{2m}\hat{p}^2_{\mu_0} + \hat{V}(q) \tag{8}$$

Polarizations

Let us now explore the two possible polarizations. In the q-polarization, the basis, labeled by n:

$$\langle q|n\rangle = \chi_n(q) = \delta_{q,\mu_n} \tag{9}$$

That is, the wave functions will only have support on the set γ_{μ_0}. Alternatively, one can think of a state as completely characterized by the 'Fourier coefficients' a_n: $\psi(q) \leftrightarrow a_n$, which is the value that the wave function $\psi(q)$ takes at the point $q = \mu_n = n\,\mu_0$. Thus, the Hamiltonian is a difference equation when acting on $\psi(q)$.

Solving the time independent Schrödinger equation $\hat{H} \cdot \psi = E\,\psi$ amounts to solving the difference equation for the coefficients a_n.

In the momentum polarization, the operator $\hat{p}^2_{\mu_0}$ acts as a multiplication operator:

$$\hat{p}^2_{\mu_0} \cdot \psi(p) = \frac{2\hbar^2}{\mu_0^2} \left[1 - \cos\left(\frac{\mu_0\, p}{\hbar}\right)\right] \psi(p) \tag{10}$$

The operator $\hat{q}$ is a derivative:

$$\hat{q} \cdot \psi(p) := -i\hbar\, \partial_p\, \psi(p). \tag{11}$$

A generic potential $V(q)$ has to be defined by means of spectral theory defined now on a circle.

Why on a circle? By restricting to a regular graph γ_{μ_0}, the functions of p that preserve it (when acting as shift operators) are of the form $e^{(i\, m\, \mu_0\, p/\hbar)}$ for m integer.

That is, what we have are the Fourier modes, labeled by m, of period $2\pi\hbar/\mu_0$ in the now periodic coordinate p.

Can we pretend that the variable p is now compactified? Yes!

The inner product on periodic functions $\psi_{\mu_0}(p)$ of p coming from the full Hilbert space $\mathcal{H}_{\rm poly}$ and given by:

$$\langle\phi(p)|\psi(p)\rangle_{\rm poly} = \lim_{L\mapsto\infty}\frac{1}{2L}\int_{-L}^{L}{\rm d}p\,\overline{\phi}(p)\,\psi(p)$$

is equivalent to the inner product on the circle:

$$\langle\phi(p)|\psi(p)\rangle_{\mu_0} = \frac{\mu_0}{2\pi\hbar}\int_{-\pi\hbar/\mu_0}^{\pi\hbar/\mu_0}{\rm d}p\,\overline{\phi}(p)\,\psi(p)$$

with $p\in(-\pi\hbar/\mu_0,\pi\hbar/\mu_0)$.

Remark:

As long as one restrict attention to the graph γ_{μ_0}, one can work in this separable Hilbert space $\mathcal{H}_{\gamma_{\mu_0}}$ of square integrable functions on S^1. Immediately, one can see the limitations (or not?) of this description:

i) If the mechanical system has complete orbits for which this approximation is valid, then one could expect that both the *effective* classical and the polymeric quantum descriptions should approximate the standard one.

ii) If, on the other hand, the mechanical system to be quantized is such that its orbits have values of the momenta p that are not small compared with $\pi\hbar/\mu_0$ then the approximation taken will be very poor, and we don't expect neither the effective classical description nor its quantization to be close to the standard one.

3. Continuum Limit

The dynamical theory as presently defined has a regulator $\mu_0 \neq 0$. Intuitively we expect that in the limit $\mu_0 \longrightarrow 0$ we recover the 'continuum theory'.

Do we?

But exactly what does this mean? If we just make μ_0 smaller we change the quantum theory, but in what sense can be sure that we are approaching 'the' continuum theory.

If we just lie in $\mathcal{H}_{\text{poly}}$ and refine the lattice, in the limit we would get a state as:

$$\Psi(q) = \sum_{\mu} \Psi(\mu)\, \chi_{\mu}(q)$$

where the sum is over a continuous parameter μ. Its associated norm in $\mathcal{H}_{\text{poly},x}$

is:

$$|\Psi(q)|^2_{\text{poly}} = \sum_{\mu} |\Psi(\mu)|^2 \to \infty$$

which blows up.

This does not work.

Next idea?

We can define the notion of a scale defined by regular decomposition C_n, with intervals of length $\mu_n = \mu_0/2^n$, and approximate functions on the continuum by step functions.

That is, we define an embedding, for each scale C_m from $\mathcal{H}_{\text{poly}}$ to $\mathcal{H}_{\text{Schr}}$ by means of a step function:

$$\sum_n \Psi(n\mu_m)\, \chi_{n\mu_m}(q) \to \sum_n \Psi(n\mu_m)\, \chi_{\alpha_n}(q) \ \in \ \mathcal{H}_{\text{Schr}}$$

with $\chi_{\alpha_n}(q)$ a characteristic function on the interval $\alpha_n = (n\mu_m, (n+1)\mu_m)$.

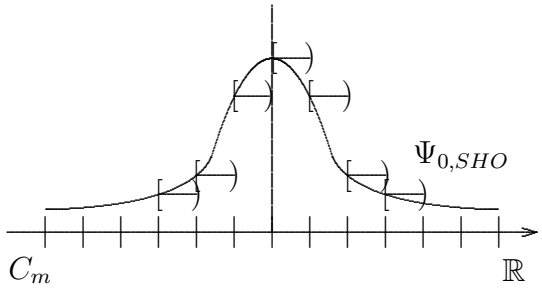

Figure 1: The solid continuous line represents the graph of a Schrödinger state $\Psi_{0,\mathrm{Sch}}$ and the piecewise constant function represents $\Psi^{\mathrm{ren}}_{0,C_m}$.

We have to define the limit of the Hamiltonian 'at scale' C_m, and for Hamiltonians of the form $H = p^2 + V(q)$ it *can* be defined.

Let us denote the kinematic polymeric Hilbert space at the scale C_n as $\mathcal{H}_{C_n}$, and his basis elements as e_{α_i,C_n}, where $\alpha_i = [ia_n, (i+1)a_n) \in C_n$. By construction this basis is orthonormal. The basis elements in the dual Hilbert space $\mathcal{H}^*_{C_n}$ we denote by ω_{α_i,C_n}. They are also orthonormal. We define $d^*_{m,n} : \mathcal{H}^*_{C_n} \to \mathcal{H}^*_{C_m}$

as the 'coarse graining' map between the dual Hilbert spaces, that sends the part of the elements of the dual basis to zero while keeping the information of the rest: $d^*_{m,n}(\omega_{\alpha_i,C_n}) = \omega_{\beta_j,C_m}$ if $i = j2^{n-m}$, in the opposite case $d^*_{m,n}(\omega_{\alpha_i,C_n}) = 0$. We define the quadratic form $h_n : \mathcal{H}_{C_n} \to \mathbb{R}$, given by $h_n(\psi) = \lambda^2_{C_n}(\psi, H_n\psi)$, where $\lambda^2_{C_n}$ is a normalizaton factor. We will see later that this rescaling of the inner product is necessary in order to guarantee the convergency of the renormalized theory. The completely renormalized theory at this scale is obtained as

$$h^{\rm ren}_m := \lim_{C_n \to \mathbb{R}} d^\star_{m,n} h_n. \tag{12}$$

and the renormalized Hamiltonians are compatible with each other, in the sense that

$$d^\star_{m,n} h^{\rm ren}_n = h^{\rm ren}_m.$$

In order to analyze the conditions for the convergence in (12) let us express the Hamiltonian in terms of its eigen covectors end eigen values. We will work with effective Hamiltonians that have a purely discrete spectrum (labelled by

ν) $h_n \cdot \Psi_{\nu,C_n} = E_{\nu,C_n} \Psi_{\nu,C_n}$. Thus, we can write

$$h_m^{\nu_{\text{cut-off}}} = \sum_{\nu=0}^{\nu_{\text{cut-off}}} E_{\nu,C_m} \Psi_{\nu,C_m} \otimes \Psi_{\nu,C_m} \,, \tag{13}$$

where the eigen covectors $\Psi_{\nu,C_m} \in \mathcal{H}^\star_{C_m}$ are normalized according to the inner product rescaled by $\frac{1}{\lambda^2_{C_n}}$, and the cut-off can vary up to a scale dependent bound, $\nu_{\text{cut-off}} \leq \nu_{\max}(C_m)$.
In the presence of a cut-off, the convergence of the microscopically corrected Hamiltonians, equation (12) is equivalent to the existence of the following two limits. The first one is the convergence of the energy levels,

$$\lim_{C_n \to \mathbb{R}} E_{\nu,C_n} = E^{\text{ren}}_\nu \,. \tag{14}$$

Second is the existence of the completely renormalized eigen covectors,

$$\lim_{C_n \to \mathbb{R}} d^\star_{m,n} \Psi_{\nu,C_n} = \Psi^{\text{ren}}_{\nu,C_m} \in \mathcal{H}^\star_{C_m} \subset Cyl^\star_x \,. \tag{15}$$

When the completely renormalized eigen covectors exist, they form a collection that is $d^\star$-compatible.
The collection of the $d^\star$-compatible Hamiltonians can be extended to act on $\mathcal{H}_{\mathrm{poly},x}$. In that case we can define the Hamiltonian (with cut-off) $h_{\mathbb{R}}^{\nu_{\mathrm{cut-off}}\mathrm{ren}} : \mathcal{H}_{\mathrm{poly},x} \to \mathbb{R}$ in the continuum as

$$h_{\mathbb{R}}^{\nu_{\mathrm{cut-off}}\mathrm{ren}}(\delta_{x_0,q}) := \lim_{C_n \to \mathbb{R}} h_n^{\nu_{\mathrm{cut-off}}\mathrm{ren}}([\delta_{x_0,q}]_{C_n}), \tag{16}$$

where x_0 is a vertex of one of the intervals at some scale C_n.
The normalization factors that lead to the convergences needed is $\lambda_{C_n}^2 = 2^n$, so that the renormalized inner product in $\mathcal{H}_n^\star$ is

$$(\omega_{\alpha_i}, \omega_{\alpha_j})_{C_n}^{\mathrm{ren}} = \frac{1}{2^n}\delta_{ij}. \tag{17}$$

Let's call $\mathcal{H}_{C_n}^{\star\mathrm{ren}}$ the resulting Hilbert space of covectors.
The sequence of $d^\star$-compatible normalizable covectors define an element of $\overleftarrow{\mathcal{H}}_{\mathbb{R}}^{\star\mathrm{ren}}$, which is the projective limit of the renormalized spaces of covectors

$$\overleftarrow{\mathcal{H}}{}^{\star\mathrm{ren}}_{\mathbb{R}} = \varprojlim_{C_n \to \mathbb{R}} \mathcal{H}^{\star\mathrm{ren}}_{C_n}. \tag{18}$$

The inner product in this space is defined by

$$(\{\Psi_{C_n}\}, \{\Phi_{C_n}\})^{\mathrm{ren}}_{\mathbb{R}} := \lim_{C_n \to \mathbb{R}} (\Psi_{C_n}, \Phi_{C_n})^{\mathrm{ren}}_{C_n}.$$

Since the inner product is degenerate, the physical Hilbert space is defined as

$$\mathcal{H}^{\star}_{\mathrm{phys}} := \overleftarrow{\mathcal{H}}{}^{\star\mathrm{ren}}_{\mathbb{R}} \,/\, \ker(\cdot,\cdot)^{\mathrm{ren}}_{\mathbb{R}}$$

$$\mathcal{H}_{\mathrm{phys}} := \mathcal{H}^{\star\star}_{\mathrm{phys}}$$

There exists an unitary anti-isomorphism from the Hilbert space in the Schrödinger representation $L^2(\mathbb{R}, \mathrm{d}x)$ to $\mathcal{H}_{\mathrm{phys}}$. The set of the completely renormalized proper covectors of the effective theories converge as $C_n \to \mathbb{R}$ to a completely renormalized proper covector in the continuum. This covector when acting on the basis of $\mathcal{H}_{\mathrm{poly},x}$ produces the shadow of the Schrödinger wave functions. Let's see an example:

4. Simple Harmonic Oscillator

In this part, let us consider the example of a Simple Harmonic Oscillator (SHO) with parameters m and ω, classically described by the following Hamiltonian

$$H = \frac{1}{2m}\, p^2 + \frac{1}{2}\, m\, \omega^2\, q^2.$$

Recall that from these parameters one can define a length scale $D = \sqrt{\hbar/m\omega}$. In the standard treatment one uses this scale to define a complex structure J_D (and an inner product from it), as we have described in detail that uniquely selects the standard Schrödinger representation.

At scale C_n we have an effective Hamiltonian for the Simple Harmonic Oscillator (SHO) given by

$$H_{C_n} = \frac{\hbar^2}{m a_n^2}\left[1 - \cos\frac{a_n p}{\hbar}\right] + \frac{1}{2}\, m\, \omega^2 q^2\,. \tag{19}$$

If we interchange position and momentum, this Hamiltonian is exactly that of a pendulum of mass m, length l and subject to a constant gravitational field g:

$$\hat{H}_{C_n} = -\frac{\hbar^2}{2ml^2}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + mgl(1-\cos\theta)$$

where those quantities are related to our system by,

$$l = \frac{\hbar}{m\,\omega\,a_n}\,, \quad g = \frac{\hbar\,\omega}{m\,a_n}\,, \quad \theta = \frac{p\,a_n}{\hbar}$$

That is, we are approximating, for each scale C_n the SHO by a pendulum. The quantum system will have a spectrum for the energy that has two different asymptotic behaviors, the SHO for low energies and the planar rotor in the higher end. As we refine our scale and both the length of the pendulum and the height of the periodic potential increase, we expect to have an increasing number of oscillating states (for a given pendulum system, there is only a finite number of such states).

The relevant question is whether the conditions for the continuum limit to exist are satisfied. This question has been answered in the affirmative in `gr-qc/0610072`: The eigen-vectors and eigen functions of the discrete systems, which represent a discrete and non-degenerate set, approximate those of the continuum, namely, of the standard harmonic oscillator. For

$$H_{C_n} \cdot \Psi_{\nu,n} = E_{\nu,n}\, \Psi_{\nu,n},$$

we have

$$\Psi_{\nu,n} \to \Psi_{\nu,\mathrm{SHO}}$$

for $n \to \infty$, and for all ν labeling the excited levels. Also,

$$E_{\nu,n} \to E_{\nu,\mathrm{SHO}}$$

In the sense explained before. This convergence implies that the continuum limit exists as we understand it.

What happens then to other systems such as a free particle?

4. Free Polymer Particle

The Hamiltonian of a free particle and the corresponding time independent Schrödinger equation, in the $p-$polarization, is given by

$$\left[\frac{\hbar^2}{ma_n^2}(1 - \cos\frac{a_n p}{\hbar}) - E_{C_n} \right] \tilde{\psi}(p) = 0$$

where we now have that $p \in S^1$, with $p \in (-\frac{\pi\hbar}{a_n}, \frac{\pi\hbar}{a_n})$.
In this case the spectrum is continuum. There is an upper bound in the value of the energy: $E_{\max} = 2\hbar^2/ma_n^2$.

For the polymer free particle we have,

$$\tilde{\psi}_{C_n}(p) = c_1\delta(p - P_{C_n}) + c_2\delta(p + P_{C_n})$$

The inverse Fourier transform yields, in the 'x representation',

$$\psi_{C_n}(x_j) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\hbar/a_n}^{\pi\hbar/a_n} \tilde{\psi}(p)\, e^{\frac{i\,a_n}{\hbar}\,p\,j}\, \mathrm{d}p = \frac{\hbar\sqrt{2\pi}}{a_n} \left(c_1 e^{i x_j P_{C_n}/\hbar} + c_2 e^{-i x_j P_{C_n}/\hbar} \right).$$

with $x_j = a_n\, j$ for $j \in \mathbb{Z}$.

In the limit $n \to \infty$, $E_{C_n} \to E = p^2/2m$ so we can be certain that the eigenvalues for the energy converge (when fixing the value of p). The covectors are $\Psi_{C_n} = (\psi_{C_n}, \cdot)^{\mathrm{ren}}_{C_n} \in \mathcal{H}^{\star\mathrm{ren}}_{C_n}$. Then we can bring microscopic corrections to scale C_m and look for convergence of such corrections

$$\Psi^{\mathrm{ren}}_{C_m} \doteq \lim_{n\to\infty} d^\star \Psi_{C_n}.$$

It is easy to see that given any basis vector $e_{\alpha_i} \in \mathcal{H}_{C_m}$ the following limit

$$\Psi^{\mathrm{ren}}_{C_m}(e_{\alpha_i}) = \lim_{C_n\to\infty} \Psi_{C_n}(d(e_{\alpha_i}))$$

exists and thus ansures that we do recover the standard theory.

5. Polymer Quantum Cosmology

The is a mass-less scalar field φ. The metric is of the form:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

In terms of the coordinates $(a, p_a, \varphi, p_\varphi)$ for the phase space Γ of the theory, all the dynamics is captured in the Hamiltonian constraint

$$\mathcal{C} := -\frac{3}{8}\,\frac{p_a^2}{|a|} + 8\pi G\,\frac{p_\varphi^2}{2|a|^3} \approx 0$$

One can rewrite the equation as:

$$\frac{3}{8}\,p_a^2\,a^2 = 8\pi G\,\frac{p_\varphi^2}{2}$$

At his point we have the freedom in choosing the variable that will be quantized and the variable that will not be well defined in the polymer representation. The

standard choice is that p_a is not well defined and thus, a and any geometrical quantity derived from it, is quantized.

Let us now consider this polymer description from the perspective of its effective classical theory. Let us replace $p_a \mapsto \sin(\lambda\, p_a)/\lambda$. The effective classical Hamiltonian constraint is:

$$\mathcal{C}_\lambda := -\frac{3}{8}\,\frac{\sin(\lambda\, p_a)^2}{\lambda^2|a|} + 8\pi G\,\frac{p_\varphi^2}{2|a|^3} \approx 0$$

We can compute effective equations of motion by means of the equations: $\dot{F} := \{F, \mathcal{C}_\lambda\}$, for any observable $F \in C^\infty(\Gamma)$, and where we use the effective action:

$$S_\lambda = \int \mathrm{d}\tau (p_a\,\dot{a} + p_\varphi\,\dot{\varphi} - N\,\mathcal{C}_\lambda)$$

with the choice $N = 1$.
For the WdW, if $\dot{a} < 0$ initially, it will remain negative and the universe collapses, reaching the singularity in a finite proper time.

In the previous cases, one could have classical trajectories that remained, for a given choice of parameter λ, within the region where $\sin(\lambda p)/\lambda$ *is* a good approximation to p.

But in PQC, for any value of p_φ (that uniquely fixes the trajectory in the (a, p_a) plane), there *will* be a bounce: *Every* classical trajectory will pass from a region where the approximation is good to a region where it is not; this is where the 'quantum corrections' kick in and the universes bounces.

We shall have wave functions $\Psi(a, \varphi)$. With this choice and a particular factor ordering we have,

$$\left[\left(\frac{1}{\lambda}\sin(\lambda a)\frac{\partial}{\partial_a}\right)^2 + \frac{32\pi}{3}\,\ell_p^2\,\frac{\partial^2}{\partial\varphi^2}\right]\cdot\Psi(a,\varphi) = 0$$

as the Polymer Wheeler-DeWitt equation.

Summary

- One can obtain the kinematical structure of the polymer representation as a limit of Schrödinger (Dynamics?)
- The continum limit can be defined for 'well defined' systems.
- Examples of these sytems are given by the harmonic oscillator and the free particle.
- Polymer quantum cosmology has a classical behavior radically different. Continumm limit becomes much more subtle!
- Stay tuned!

BIBLIOGRAPHY

More details can be found in:

gr-qc/0610072

and

gr-qc/07-SOON