# Perturbative and non-perturbative treatments of 2+1 gravity coupled to scalars

## SUMMARY

- **Introduction**
- **Einstein-Rosen waves**
- **Perturbative vs. non-perturbative aspects of quantization**
  - ❶ **Perturbative vs. non perturbative Fock spaces**
  - ❷ **Coupling constant expansions**

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# INTRODUCTION

- 1-(spacelike) Killing vector reductions of gravity in 3+1 dimensions can be interpreted as theories of **2+1 gravity** coupled to **massless scalar matter**.
- If we require the existence of **another spacelike Killing** field these models simplify further
  - **Non polarized case:** If the Killings are not hypersurface orthogonal then the model is equivalent to a **sigma model** in **1+1 dimensions** with **two massless, coupled, scalar fields** $\rightsquigarrow$ **perturbative treatments** (Niedermaier,...).
  - **Polarized case:** If both Killings are **hypersurface orthogonal** [$\Rightarrow$ the Killing vector fields are, themselves, orthogonal]. An important example are Einstein-Rosen waves $\Leftrightarrow$ axisymmetric **2+1 gravity** coupled to a **massless scalar field** in a **fixed, auxiliary, Minkowskian background** $\rightsquigarrow$ **exact Fock space quantization** (Ashtekar, Pierri, Varadarajan,...).

✷There is an interesting relationship between symmetry reductions of 3+1 gravity and lower dimensional theories of gravity coupled to massless matter fields

✷**Purpose of the talk:** Discuss issues related to **perturbative** and **non-perturbative** treatments of quantum gravity from the detailed study of quantized Einstein-Rosen waves.

| Does the non-renormalizability of General Relativity in the usual perturbative formulation mean that it cannot be consistently quantized? |
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❶ Inequivalent quantizations: **Perturbations** around Minkowski vs. full **non-perturbative** quantization.

❷ Quantized Einstein-Rosen waves and **perturbative expansions** in the effective gravitational constant $G$.

**FACTS, QUESTIONS, AND COMMENTS**

# LINEARLY POLARIZED CYLINDRICAL GRAVITATIONAL WAVES

✷ Some symmetry reductions of GR can be exactly quantized despite having **local degrees of freedom** and (some) **diff-invariance**.

✷ The dynamics is given by the following Hamiltonian [Ashtekar, Varadarajan, Phys.Rev. D50 (1994) 4944, `gr-qc/9406040`].

$$H = \frac{1}{4G}(1 - e^{-4GH_0})$$
$$H_0 = \frac{1}{2}\int_0^\infty \mathrm{d}r\, r[(\partial_t\phi_g)^2 + (\partial_r\phi_g)^2]$$

Here $c = \hbar = 1$, $G = \hbar G_3$, and $G_3$ is the gravitational constant per unit length in the direction of the symmetry axis. $G$ plays the role of a **characteristic length** of the system (analogous to the Planck length).

✷The physical Hamiltonian is non-trivial because it is a **function** of the free Hamiltonian for a massless, axisymmetric, scalar field evolving in a 2+1 dimensional Minkowskian background.

# QUANTIZATION

- Fock space quantization is possible [Ashtekar, Pierri, J.Math.Phys. 37 (1996) 6250, `gr-qc/9606085`]

- The **quantum field** $\hat{\phi}_g(r)$ can be written in terms of creation and annihilation operators $\hat{A}(k)$ and $\hat{A}^\dagger(k)$ (satisfying the usual commutation relations) as

$$\hat{\phi}_g(r) = \sqrt{4G}\int_0^\infty \mathrm{d}r\, J_0(kr)[\hat{A}(k) + \hat{A}^\dagger(k)],$$

- The **quantum Hamiltonian** is (normal ordering the exponent)

$$\frac{1}{4G}\left\{1 - \exp\left[-4G\int_0^\infty \mathrm{d}k\, k\hat{A}^\dagger(k)\hat{A}(k)\right]\right\}$$

and the **unitary evolution operator**

$$\hat{U}(t_2, t_1) = \exp\left[i\frac{(t_2 - t_1)}{4G}\left\{1 - \exp\left[-4G\int_0^\infty \mathrm{d}k\, k\hat{A}^\dagger(k)\hat{A}(k)\right]\right\}\right]$$

## Comments:

- **This is it** as far as the quantization is concerned.
- We can compute **exactly** the evolution of any Fock space state that we wish.
- **We know the $S$ matrix** because it can be obtained from $\hat{U}(t_2, t_1)$ as the limit

$$S = \lim_{t_2 \to \infty, t_1 \to -\infty} \hat{U}(t_2, t_1)$$

# What does perturbative mean?

✷ In the usual **perturbative** approach to quantum gravity (i.e. 't Hooft & Veltman, Goroff & Sagnotti), the main goal is to obtain counterterms for the Einstein-Hilbert action to find out if the theory is renormalizable. Eventually one would like to answer physical questions, compute $S$-matrix elements,...

✷ A clever way to do it (imported from the perturbative treatment of Yang-Mills) is to use the **background field method**. For gravity this amounts to expanding $g_{ab} = g^0_{ab} + h_{ab}$ where $g^0_{ab}$ is a **fixed, non-singular, background metric** and $h_{ab}$ is a "quantum fluctuation" (that appears, for example in the path integral measure). In a sense this is just a change of variables in the functional integral.

✷ Here the metric $g^0_{ab}$ plays **two roles**: As the background field in terms of which the (covariant) counterterms are built and as a convenient extra structure useful to define propagators, introduce causality,...

- If the Minkowski metric $\eta_{ab}$ is chosen as background one cannot write the counterterms in terms of it because they involve the curvature that vanishes for $\eta_{ab}$. The counterterms would appear as powers of $h_{ab}$ that can be collected in covariant counterterms.
- In any case the Lagrangian density $\sqrt{|g|}R$ is substituted by an **infinite series** involving arbitrary powers of $h_{ab}$. This series defines the **vertices** needed to construct Feynman diagrams.
- Feynman diagrams are used to write down **Green's functions as power series** in the coupling constant $G_N$.
- There are **two meanings** of the word "perturbative" here:
  - The one referring to the expansion $g_{ab} = g^0_{ab} + h_{ab}$.
  - The one referring to expansions of physical quantities (Green functions, $S$ matrix,...) as powers of $G_N$.
- Both are relevant to the quantization of Einstein-Rosen waves.

# First meaning

[F. B., G. Mena, E. Villaseñor, Phys. Rev. D70 (2004) `gr-qc/044028`]

**A question:** What is the relation between the exact Fock quantization of ER waves and the one obtained by describing the system as a "perturbation" of Minkowski?

✷ Usual way to write the metric for ER waves (starting point for the usual treatments and the AP quantization).

$$ds^2 = e^{-\psi}[-N^2 dt^2 + e^{\gamma}(dR + N^R dt)^2 + r^2 d\theta^2] + e^{\psi} dZ^2$$

✷ By introducing the **new fields** $\bar{N}$, $\bar{N}^R$, $\bar{\gamma}$, $\bar{\psi}$, and $\bar{\rho}$ defined as

$$\gamma := \log[(1+\bar{\psi})(1+\gamma \bar{-} \bar{\psi})] \qquad \psi := \log(1+\bar{\psi})$$

$$r := \sqrt{(1+\bar{\psi})(R^2(1-\bar{\psi}) + R\bar{\rho})}$$

$$N := \sqrt{(1+\bar{\psi})(1+2\bar{N} - \bar{\psi} - \frac{(\bar{N}^R)^2}{1+\bar{\gamma}-\bar{\psi}})} \qquad N^R := \frac{\bar{N}^R}{1+\bar{\gamma}-\bar{\psi}}$$

the metric becomes

$$ds^2 = -(1 - 2\bar{N} - \bar{\psi})dt^2 + 2\bar{N}^R dt dR + (1 + \bar{\gamma} - \bar{\psi})dR^2$$
$$+(R^2 - R^2\bar{\psi} + 2\bar{\rho})d\theta^2 + (1 + \bar{\psi})dZ^2$$

- when [$\bar{N}$, $\bar{N}^R$, $\bar{\gamma}$, $\bar{\psi}$, and $\bar{\rho}$] vanish we recover the Minkowski metric.

- One can obtain the reduced Hamiltonians derived from both formulations after a suitable gauge fixing (in fact the same one in terms of the two sets of variables). There is a **canonical transformation** that relates both.

- If one considers the **linearization** of the ER model around Minkowski (by keeping only quadratic terms in the barred fields in the reduced Hamiltonian) one can get the reduced Hamiltonian for the model in linearized gravity. This turns out to be the Hamiltonian of a **massless scalar field with rotational symmetry in a 2+1 Minkowskian background**.

✷ A comment on gauge fixing. The usual gauge fixing used by Ashtekar and Pierri is **a generalization of the usual de Donder gauge** $[\bar{h}^{\nu}{}_{\mu,\nu} = 0, \bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu}]$

✷ The perturbative quantization of the model is done in the Fock space provided by the **free theory** corresponding to the linearization of the ER model. Of course one has to incorporate the interactions provided by the higher order terms in the action and use some approximation scheme to compute physical objects.

✷ **A question:** How are the Fock spaces (in particular the vacua) of the AP treatment of the model and the linearized model related?

✷ It is possible to write down the creation and annihilation variables of one of the approaches in terms of the ones for the other because we know how both formulations are related. The formulas are very complicated. One important feature: the AP creation-annihilation variables appear multiplied by $\sqrt{G}$ in these formulas so it is natural to consider expansions in powers of $\sqrt{G}$.

✷ Main result: If the perturbative vacuum is related to the AP one then it cannot be expanded as a power series of the form

$$|0\rangle_{pert} = |0\rangle_{AP} + \sum_{n=1}^{\infty} G^{n/2} |\Phi_n\rangle_{AP}$$

This is **strong indication that both quantizations are not equivalent.**

**Warning and morale:** Depending on the choice of variables that one uses to quantize the model (and certainly when one uses **perturbative approaches in this first sense**) one may end up with **non-equivalent quantizations!**

# SECOND MEANING: EXPANDING IN $G$

[F. B., G. Mena, E. Villaseñor, J.Math.Phys. 46 (2005) 062306, `gr-qc/0412028`]

✷ N-point functions (v.e.v. of products of field operators at different spacetime points) are the objects in terms of which the $S$ matrix is written in traditional approaches to Quantum Field Theory (LSZ formalism).

✷ In the present example we have the **exact** quantum evolution operator. This allows us to get the Heisenberg picture evolution of the field operators **in closed form** and write down the **exact N-point functions**.

✷ An example: the **two-point function**

$$\langle 0|\hat{\phi}_g(R_2;t_2)\hat{\phi}_g(R_1;t_1)|0\rangle =$$

$$4G\int_0^\infty J_0(R_1k)J_0(R_2k)\langle 0|\hat{A}(k;t_2)\hat{A}^\dagger(k;t_1)|0\rangle\,\mathrm{d}k =$$
$$4G\int_0^\infty J_0(R_1k)J_0(R_2k)\exp[-i(t_2-t_1)E(k)]\,\mathrm{d}k$$

Here $E(k) := \frac{1}{4G}(1-e^{-4Gk})$.

- This integral is (conditionally) **convergent**. If we introduce a momentum cutoff $\Lambda_k$ to regularize the fields it would show up as the upper integration limit. The limit $\Lambda_k \to \infty$ is well defined.
- N-point functions are obtained in a similar way (and seem to be all finite).
- In the usual perturbation schemes (for example in standard QED) one makes the ansatz that the N-point functions can be expanded as **power series** in the coupling constant ($\alpha$ for QED). The coefficients in these expansions are then obtained through the computation of **Feynman diagrams**.
- **A question:** Can we obtain such a power series expansion in terms of $G$ for the previous two-point function?
- Formal expansion

$$4G\int_0^{\infty} \mathrm{d}k J_0(R_1k)J_0(R_2k)e^{-i(t_2-t_1)k}\sum_{n=0}^{\infty} G^n P_{n+1}(k,T)$$

where $P_{n+1}(k,T)$ is a polynomial in $k$ and $T := (t_2-t_1)$ of degree $n+1$ in $k$.

- We see that each term in the $G$ Taylor expansion is a very bad behaved integral in $k$.
- If one introduces a **cutoff** one can find conditions that guarantee **uniform convergence**. However they are rather restrictive, involve $G$, and it is very unlikely that they will be universally valid for arbitrary N-point functions.
- Notice that $G$ can be seen both as a **fixed length scale** or as a **coupling constant**, this leads to two possible points of view: either one considers a **weak coupling limit** or looks at the system at typical distances much larger than $G$.
- Several possible attitudes:
  - Keep the cut-off and try to find a way to remove it by absorbing it in a **renormalized** $G$ (interpreted as a coupling constant).
  - Use *ad-hoc* methods to study the limiting behavior of Green functions for small values of $G$ or, equivalently, for $R_1, R_2, t_1, t_2 \gg G$ (asymptotic analysis).

# Asymptotic expansions of two-point functions

✷ Not easy to implement (basically all the standard methods involving steepest descents, saddle point, Mellin transforms,... **fail**).

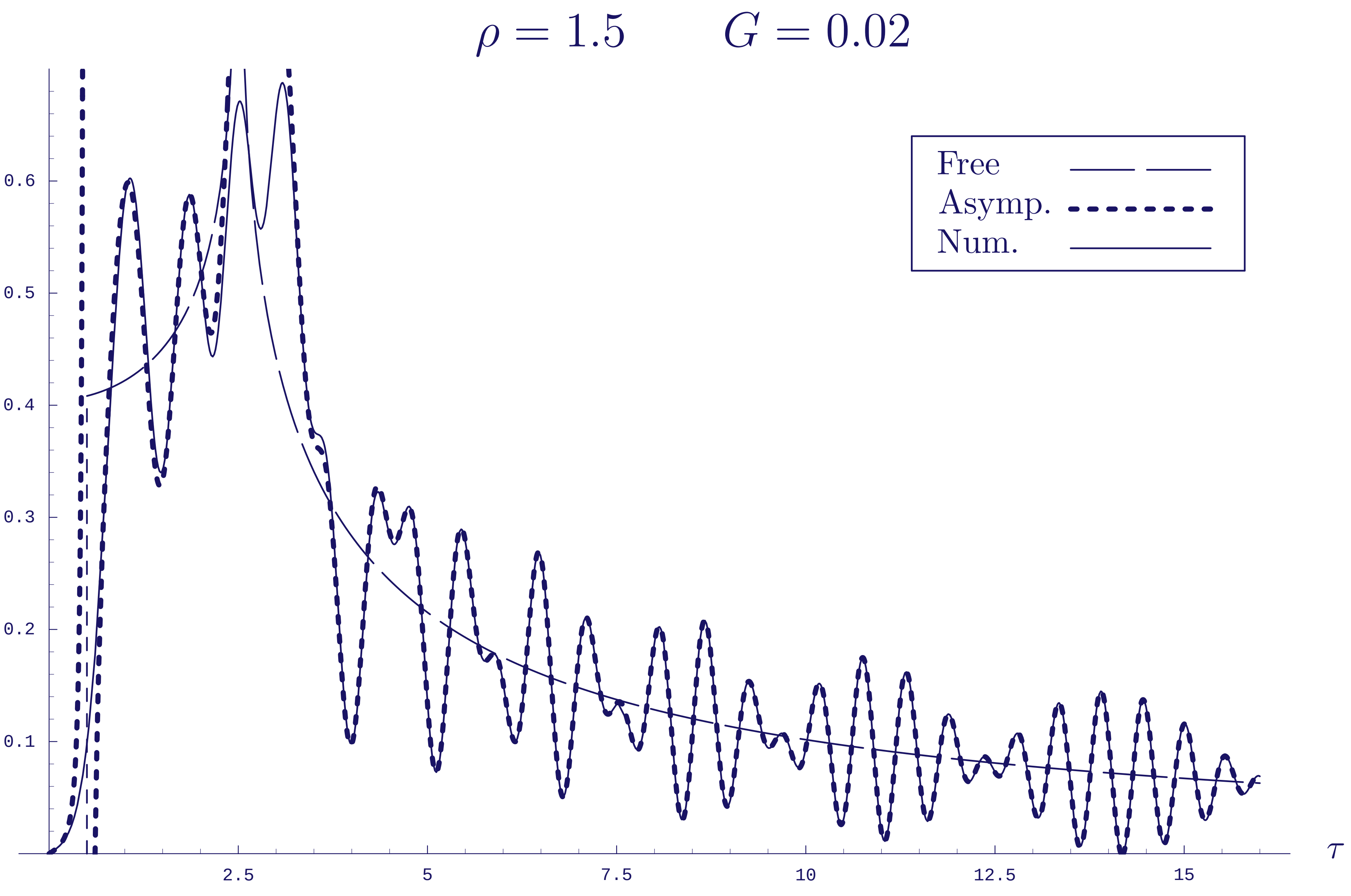
✷ At the end of the day it can be done. The result is complicated. Just as an example let us look at the asymptotic value of the imaginary part of the two point function (valid when $|R_2 - R_1| < t_2 - t_1 < |R_2 + R_1|$ and in the limit $G \longrightarrow 0$)

$$\frac{1}{\sqrt{R_1 R_2}} K\left(\sqrt{\frac{(t_2-t_1)^2-(R_2-R_1)^2}{4R_1R_2}}\right)$$
$$+\sqrt{2G}\,\Im\left\{\frac{e^{-i\frac{\pi}{4}} e^{\frac{i}{4G}[t_2-t_1-|R_2-R_1|(1+\log\frac{t_2-t_1}{|R_2-R_1|})]}}{\sqrt{\pi R_1 R_2 |R_2-R_1|}\log\frac{t_2-t_1}{|R_2-R_1|}}\right\}+\cdots$$

Here $K(k)$ denotes the complete elliptic integral of the first kind.

✷ There is a term independent of $G$ and a first, **non-analytic**, term that appears in a systematic expansion.

In case you cannot believe that the real behavior is given by the previous expression let me show you a plot [here $\rho := R_2/R_1$, $\tau := (t_2 - t_1)/(4G)$].



## Comments:

- An *a priory* choice of perturbative behavior (Taylor expansion in $G$) is not suitable to capture the physics of the system.
- The theory is well defined but seems to be **intrinsically non-perturbative**.
- It is not clear how a renormalization scheme (by redefining $G$ to absorb quantities that diverge when the cutoff goes to infinity) would work because the final result is not a power series.
- A renormalization scheme would end up giving $G(k;\mu)$ [$\mu$ would be the arbitrary scale that must be introduced in the renormalization process]. Having in mind that $G$ is the characteristic length scale of the model, does it make sense to consider it **momentum dependent?**
- The model is, in a clear sense, **finite**. However usual finite QFT's (extended supersymmetry, strings,...) have finite coefficients for each power of the coupling constants which means that there is no beta function and no running coupling constants (only the bare ones).

✷ This is not true in our case if one chooses to expand in integer powers of $G$. Can we still say that the beta function is zero and there is no running coupling constant?

✷ I believe that the model is **neither renormalizable nor finite** in the usual perturbative sense even though it is perfectly well defined.

**In my opinion this may be the case for full quantum gravity.**